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THE JUNE MEETING IN SEATTLE

The two hundred ninety-first regular meeting of the Society was held at the University of Washington, Seattle, Washington, on Saturday, June 13, 1931. The meeting was called to order at 10:30 a.m. by Professor R. E. Moritz. The meeting was presided over later by Professor W. E. Milne.

The attendance included the following eighteen members of the Society:

C. H. Ashton, C. R. Ballantine, J. P. Ballantine, John Biggerstaff, A. F. Carpenter, C. M. Cramlet, G. I. Gavett, A. R. Jerbert, L. H. McFarlan, C. E. Magnusson, W. E. Milne, R. E. Moritz, Hermance Mullemeister, L. I. Neikirk, F. S. Nowlan, E. D. Pepper, J. R. Vatnsdal, R. M. Winger.

In view of the summer meeting at Los Angeles in 1932, it was voted to recommend to the Council that the June meeting in the Northwest be omitted next year.

The visitors were guests of the local members at a luncheon in the University commons, followed by a boat trip through the Government locks to Bainbridge Island, where a picnic supper was served, Mr. and Mrs. E. Schuyten courteously opening their summer home for the occasion.

The titles of papers read at the meeting follow. Those whose abstract numbers are followed by the letter *t* were read by title. Mr. Ayres was introduced by Professor McFarlan. Mr. Rust was introduced by Professor Evans. Dr. Schoenberg was introduced by Professor Bliss.

1. *Integral equations and the cooling problem*, by Mr. W. M. Rust, Jr. (Abstract No. 37–5–223–*t*.)

2. *On complex methods of summability*, by Dr. R. P. Agnew. (National Research Fellow). (Abstract No. 37–5–224–*t*.)

3. *Topological invariance of sub-complexes of singularities*, by Dr. A. B. Brown. (Abstract No. 37–5–225–*t*.)

4. *The transformation C of nets in hyperspace*, by Professor V. G. Grove. (Abstract No. 37–5–226–*t*.)

5. *On universal quadratic null forms in five variables*, by Dr. R. S. Underwood. (Abstract No. 37–5–227–*t*.)

6. *On a lacuna in the theory of systems of partial differential equations*, Part II, by Mr. L. B. Robinson. (Abstract No. 37–5–228–*t*.)

7. *On representation by positive binary quadratic forms*, by Dr. Gordon Pall (National Research Fellow). (Abstract No. 37–5–229–*t*.)

8. *Magic cubes which are uniform step cubes*, by Dr. Emma W. McDonald. (Abstract No. 37–5–230–*t*.)

9. *A triad of ruled surfaces defined by reciprocal polars*, by Professor A. F. Carpenter. (Abstract No. 37–5–231.)

10. *A ternary analog of an Abelian group*, by Dr. D. H. Lehmer (National Research Follow). (Abstract No. 37–5–232–*t*.)

11. *Useful functions associated with rational cubic curves*, by Professor Clifford Bell. (Abstract No. 37–7–235–*t*.)

12. *A converse of the theorem regarding the separation of E_3 by a closed two-dimensional manifold of genus p*. Second paper, by Professor R. L. Wilder. (Abstract No. 37–7–236–*t*.)

13. *An interpretation of the invariants of a system of linear homogeneous second-order differential equations*, by Professor C. M. Cramlet. (Abstract No. 37–7–237.)

14. *On certain projective trochoids*, by Professor R. M. Winger. (Abstract No. 37–7–238.)

15. *The numerical solution of linear equations by vectors*, by Professor J. P. Ballantine. (Abstract No. 37–7–239.)

16. *A certain congruence associated with a ruled surface*, by Professor A. R. Jerbert. (Abstract No. 37–7–240.)

17. *On the numerical integration of certain differential equations of the second order*, by Professor W. E. Milne. (Abstract No. 37–7–241.)

18. *A new theory of depreciation of physical assets*, by Professor R. E. Moritz. (Abstract No. 37–7–242.)

19. *On a totally discontinuous function*, by Professor R. E. Moritz. (Abstract No. 37–7–243.)

20. *On the Moore-Kline problem*, by Dr. Leo Zippin. (National Research Fellow). (Abstract No. 37–7–244–*t*.)

21. *Some applications of the calculus of variations to Riemannian geometry*, by Dr. I. J. Schoenberg (International Research Fellow). (Abstract No. 37–7–245–*t*.)

22. *On separation, comparison, and oscillation theorems for self-adjoint systems of second-order linear differential equations*, by Professor G. A. Bliss and Dr. I. J. Schoenberg. (Abstract No. 37–7–246–*t*.)

23. *Concerning a mixed functional equation and an associated*

invariant theory, by Professor A. D. Michal. (Abstract No. 37–7–247–*t*.)

24. *Some theorems concerning matrices of continuous functions*, by Mr. H. C. Ayres. (Abstract No. 37–7–248–*t*.)

25. *The problem of minimizing a double integral with adjoined partial differential equations*, by Professor L. H. McFarlan. (Abstract No. 37–7–249.)

26. *Note on kernels of positive type*, by Dr. Caroline E. Seely. (Abstract No. 37–7–250–*t*.)

27. *Concerning continuous images of the interval*, by Professor G. T. Whyburn. (Abstract No. 37–7–251–*t*.)

28. *On direct products*, by Dr. A. A. Albert. (Abstract No. 37–5–234–*t*.)

29. *Remarks concerning a paper of W. L. Ayres*, by Professor Karl Menger. (Abstract No. 37–7–252–*t*.)

T. M. PUTNAM,
Associate Secretary

MOULTON ON DIFFERENTIAL EQUATIONS

Differential Equations. By Forest Ray Moulton. New York, Macmillan Company, 1930.

This is an important book dealing exclusively with ordinary differential equations. It could have been written only by one thoroughly familiar with the details of certain phases of mathematical physics. In fact it has a decided astronomical slant. One needs only to run over the chapter headings to be convinced of this, and a careful reading of the book confirms this preliminary conviction. It is different from any other text on the subject known to the reviewer by virtue of the happy combination it achieves of theory and applications.

Various important points of the theory are illustrated by the working out in detail of the problems of elliptic motion, the deviations of falling bodies, and the damped gyroscope. A discussion of the sine-amplitude function also furnishes illustrative material. As the author justly remarks in his preface, these problems, in respect to difficulty, "are far beyond the type usually used for illustrations," and they "help to bring out the richness of the logical results contained in the general theorems, a richness which would not be suspected by a beginner in the field."

The fact that Chapter 11 is devoted wholly to numerical solutions is a further indication of the practical nature of the book. But this work is preceded by a new and careful discussion of the logical foundations of the numerical processes described. As to the details of the numerical work, it seems to the reviewer that the discussion is too elaborate for the general reader. There can be no question but that a certain amount of these numerical details is valuable for all students. But opinions will differ as to where the line should be drawn, and it is better to err by giving too many than too few. The chapter on Linear Differential Equations with Constant Coefficients also contains important new material, but it is also more detailed and comprehensive than is desirable for the general reader.

The main part of the book ends with a chapter on equations with periodic coefficients and one on equations in infinitely many variables. Both of these chapters contain new material.

The historical sketches at the ends of the chapters contain much important information and add materially to the value of the book. There is also a good list of questions and problems with each chapter. In addition there is a series of appendices in which a number of the standard theorems of analysis which are used in the text are proved. This will be a welcome feature to many readers of the book.

We have referred to the familiarity with the practical details shown by the author. But he has not failed to give due attention to questions of theory. In fact most of the book is devoted to a presentation of the classical theory, and the reader cannot fail to be impressed by the rigor of the demonstrations and the careful attention to clearness that is obvious on every page. He will be convinced that the author has come very close to his ideal as set forth in the

preface—"It would be inexcusable to discuss general theorems of differential equations and not to treat them with the rigor and completeness of reasoning that are characteristic of analysis of the present day. The aim of the book has been to meet current standards of rigor in the discussion of every subject treated."

There is, of course, a full discussion of the current existence theorems. In addition, the dependence of the solutions upon parameters is handled in a very suggestive and satisfactory way, as is also the expansion of a solution in the neighborhood of a variable point.

In writing such a book as this it is obviously necessary, as the author observes, to omit many important topics. We note here for the benefit of the reader that he will find nothing in the book concerning the nature of solutions in the neighborhood of singular points and no discussion of general boundary problems or of linear differential systems.

There is nothing about the adjoint of a linear homogeneous differential operator or about Green's functions. Neither is there anything about the solution of linear equations by definite integrals or about the asymptotic expansion of solutions.

The number of misprints seems to be small. We list here a few that have come to notice. On page 124 the coefficient a_s should also be subject to the condition of being a real constant. The singular point referred to on line 4 of §75, page 125, should be $\pm\frac{1}{2}(-1)^{1/2}P_2$. The equation $t=T$ on line 23 of page 191 should be $t-t_0=T$. On page 209, line 5 from the bottom, r/m should be replaced by $(r-\mu)/M$, and on page 249, line 2, the inequality sign should be replaced by the equality sign. The words *the equations* should be inserted after the word *consequently* on page 48, line 6 from the bottom.

Then there are certain slight inaccuracies for which the printer is not responsible. For example, on page 103 the statement is made that "the right members of equations (30) can be expressed in power series in e." But these right members are all zero. Evidently the equations in question were in another form when this statement was written, and when the equations were changed the statement was not changed accordingly. In the figure on page 142 the point P_3 should be between Q and P_0. The inequalities (9) on page 227 are implied by inequalities (8), and not the latter by the former.

The author says on page 226 "Now let the process pass toward a limit by subdividing the whole interval for which t and the $X_i^{(k)}(t)$ remain in the region R into a greater and greater number of subintervals in such a way that the length of *each* subinterval shall approach zero." This is meaningless unless the process of subdivision is such that every subinterval preserves its identity from its first appearance on. But it is not necessary to pass toward the limit in such a particular way.

The reader now and then comes upon a statement that causes him considerable surprise. For example, on page 117 he is told that in the principal rectangle the sine-amplitude function is "real on the line $t=\frac{1}{2}(-1)^{1/2}P_2$;" and on page 121 that "With these initial conditions x and its derivatives are real *even* functions of $t-P'$." The proof on page 131 that the second of equations (45) is a consequence of the first and of the first of equations (40) is not all clear although the conclusion itself seems fairly obvious.

The argument at the middle of page 49 shows merely that

$$\begin{vmatrix} \phi_{11}(t_0)-1 & \phi_{12}(t_0) & \phi_{13}(t_0) & \cdots \\ \phi_{21}(t_0) & \phi_{22}(t_0)-1 & \phi_{23}(t_0) & \cdots \\ \vdots & \vdots & \vdots & \\ \phi_{n1}(t_0) & \phi_{n2}(t_0) & \phi_{n3}(t_0)-1 & \cdots \end{vmatrix} = 0,$$

when not all the $A_i^{(1)}$'s are zero. The values $\phi_{i,i}(t_0)=1$ $(i=1, 2, \cdots, n)$ and $\phi_{i,j}(t_0)=0$ $(i\neq j)$, given by the author, satisfy this condition, but it is not shown that there are not other possible solutions that make the determinant of the coefficients of equations (17) equal to zero.

On page 33, in his discussion of systems of equations of the first order of the form $dx_i/dt=f_i(x_j;\mu;t)$, where μ is a parameter independent of the x_j and t, the author considers one set of conditions in which he supposes, among other conditions, that the functions $f_i(x_j;\mu;t)$ are analytic in the x_j and t uniformly with respect to μ. The reader is left in uncertainty as to what the author means by uniform analyticity. If he means what one would naturally suppose, his condition seems to be stronger than it need be inasmuch as it is only the continuity of the $f_i(x_j;\mu;t)$ as functions of the x_j and t that his argument requires to be uniform with respect to the parameter μ, in addition to ordinary analyticity.

The corollary to Theorem I on page 378 seems to imply that if the function $f(x_1, x_2, \cdots, x_n;t)$ satisfies the Lipschitz condition, or possesses derivatives, then it is continuous simultaneously in the x_i uniformly with respect to t. It is clear, however, that the author did not intend to imply so much, since he cites an example on the same page with this corollary of a function for which this is not true; namely, $f(x;t)=xt/(x^2+t^2)$ when x and t are not both zero, and $f(0,0)=0$. This function is continuous with respect to x and t separately for finite values of x and t, and has derivatives with respect to x and t. But it is not continuous in x at $x=0$ uniformly with respect to t, since if it were it would be continuous at $x=0$, $t=0$ simultaneously with respect to x and t.

The statement near the middle of page 238 to the effect that the coefficients of equations (23) are constants in case the ϕ_i of equations (18) are constants is incorrect, as is shown by the simple example $dx/dt=(x-c)t$. Here $f(x,t)=(x-c)t$ and $\phi=c$. Hence $\partial f/\partial x=t$, which is not a constant. This error is repeated at the top of page 296. The argument which leads to equation (26) on page 239 is far from convincing, although it is easy to verify directly that this equation defines a particular solution of equations (23). It is not as obvious as the author implies on page 305 that, since the right member of equation (24) is an even function of t, $\phi(t)$ must be an even function of t.

But these are only a few minor spots on an otherwise brilliant sun.

W. B. FITE

DIRAC ON QUANTUM MECHANICS

The Principles of Quantum Mechanics. By P. A. M. Dirac. Oxford, The Clarendon Press, 1930. x+257 pp.

This work may be divided into two parts, the first dealing with the general physical ideas of the quantum theory, the second treating the applications. The second part amounts to a collection of the author's well known papers on quantum dynamics which have appeared within the last half-dozen years in the Proceedings of the Royal Society. The presentation has been unified, and in many respects improved, though not without exception. It deals with the following topics: Elementary Applications; Motion in a Central Field of Force; Perturbation Theory; Collision Problems; Systems Containing Several Similar Particles; Theory of Radiation; Relativity Theory of the Electron.

It is the first part of the work which forms the essentially new contribution: it provides us with the author's conception of the fundamental ideas of the new physics, ideas which have only been implied in the works which he has published hitherto.

The principal concepts are the *states* of a system, the relation of *superposition* which may exist between states, and the *observables* which furnish information concerning the system. These notions are introduced on an a priori experimental basis, and only subsequently is a symbolism introduced for their expression, and an algebra for the formulation of their properties. Lastly, the conventional statistical interpretation is given. It is observed that this is the reverse of the customary order, in which the theory of operators comes first and the physical interpretation last, or more confusing still, is sprinkled in with the mathematical theorems. Yet we feel that our author has chosen the only proper order, either for gaining insight into the physical situation, or for giving the mathematical material its appropriate position. We only wish that he had been more thorough-going in this direction.

The style is crisp and vivid, and the intuition which the author displays and, in a sense, lends to the reader, is little short of profound. But in what concerns logical clarity of exposition, the work leaves much to be desired. For example, we are told in §§2 and 3 that a system (which may consist of one indivisible particle) the past history—or "preparation"—of which has been definitely observed to be A, "can equally well be considered" to have had part of it experience the past history B, and the other part the history C, where A, B, and C are all distinct. At this point we are reassured by recalling the author's earlier statement referring to the special case of the photon: "The description which quantum mechanics allows us to give is merely a manner of speaking which is of value in helping us to deduce and to remember the results of experiments and which never leads to wrong conclusions. One should not try to give too much meaning to it." By the end of the chapter one is enabled to cast the definition of superposition into final form, and to discover that it involves in an essential manner certain statistical concepts. In the second chapter, where the algebra of states and observables is introduced, such symbols as $=, +, -, 0$, etc., are ostensibly given a physical definition, and one empha-

tically not allowing them to be handled in the ordinary manner of algebra; nevertheless, they are so handled, and one arrives at the conclusion that what was ostensibly a definition was given merely for the sake of orientation, the actual definitions being supplied only by implication. It is our opinion that the difficulties of exposition in these and many cases of the sort are not fundamental, but could have been removed with a little care and the avoidance of the double use of terms.

On the mathematical side, the most notable feature of the work is the author's use of a space of a non-countable infinity of dimensions in which to represent his physical states. This space contains Hilbert space as a subspace, together with such further elements as the δ function and its derivatives. Although the axiomatics of this space are not given, and properties belonging to Hilbert space are carried over without justification, we believe the mere introduction of this space on a physical basis to be of great interest. As to the question of whether, if the logical work required for setting this space on a sound basis were actually to be carried through, the resulting treatment would present any advantages over the corresponding treatment by means of Hilbert space, we are unable to venture an opinion.

The notation and formal ideas are those of the transformation theory, according to which linear operators are regarded as being coextensive with infinite matrices. But von Neumann* has shown that the matrix representations of any two discontinuous operators (like those of quantum mechanics) are equivalent, in a certain sense, under a unitary transformation, a fact which should give us pause in accepting the transformation theory.

We feel that the usefulness of the book would have been enhanced by supplying it with an appendix, and by giving more references. It would have been well, in particular, to have added the name of von Neumann to that of Weyl among the exceptions to the statement that the expositions of quantum mechanics in the literature fail to take an abstract point of view and are always given by means of a definite system of coordinates (see the Introduction).† Finally, we object to the coining of the words "eigenvalue" and "eigenfunction," particularly since usage provides the terms "characteristic number" and "characteristic function," as the precise rendering of the German "Eigenwert" and "Eigenfunktion."

But if the *Principles of Quantum Mechanics* has its minor mathematical defects, I hope that my position as mathematical reviewer has not allowed me to obscure the fact that we have before us a work of no ordinary character, a work which by its depth of intuition, its originality of conception, and its power of insight, constitutes a notable advance in this department of thought.

B. O. KOOPMAN

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* J. von Neumann, Journal für Mathematik, vol. 161 (1929), p. 208.

† J. von Neumann, *Wahrscheinlichkeitstheoretischer Aufbau der Quantenmechanik*, Göttinger Nachrichten, 1927, p. 245.

PAPPUS' COMMENTARY ON EUCLID

The Commentary of Pappus on Book X of Euclid's Elements. Arabic text and translation. By William Thomson, with introductory remarks, notes, and a glossary of technical terms by Gustav Junge and William Thomson. Cambridge, Mass., Harvard University Press, 1930. 294 pp.

Of all the mathematical writers of the world the one whose name has been known to the greatest number of people is doubtless Euclid. Considering the circumstances under which he composed his classic, he was the most remarkable textbook writer that ever lived. In spite of all this, however, what is probably the most scholarly part of his writings is relatively unknown, and the works of Pappus, the best of his early commentators, have come down to us only in fragments. It is therefore a matter of special interest to scholars that Mr. Thomson has given to the English-reading world an opportunity of learning more about Book X of the *Elements*—the text of which might indeed have been found in Sir Thomas Heath's well known edition—and of learning the nature of the hitherto almost unknown commentary of Pappus, who lived some six centuries after the great classic was written.

The commentary itself is preserved, so far as known, in only a single Arabic manuscript, MS. 2457 of the Bibliothèque Nationale in Paris. It was described by Woepcke in 1856 and the Arabic text was later published by him. This text was translated into German by Suter and was published in 1922, after the latter's death, evidently without consulting the original Arabic text. Woepcke had assigned the commentary to one Valens, probably the astronomer Vettius Valens, a contemporary of Ptolemy (c. 150), basing his contention upon his reading of the abridgment of the author's name (the consonants only) as *Bls*. Suter, however, sensed the fact that this was an error in reading the Arabic, and that it should be *Bbs*, and therefore stated that it might stand for Pappus. From the text, however, he felt that this was doubtful, suggesting that it might more probably have been Proclus. Heiberg (*Euklid-Studien*, p. 169), than whom we have no better authority, felt that Pappus was the author, an opinion in which Suter seems finally to have concurred (*Abhandlungen zur Geschichte der Mathematik*, vol. 10, pp. 49, 211), which met the approval of Sir Thomas Heath, and which seems fully confirmed in the present edition.

Book X of the *Elements* sums up the work of the Pythagoreans with respect to commensurable magnitudes and hence to rational and irrational lines. It was a result of the discovery of incommensurability that Eudoxus was led to write, or at any rate prepare the ground for, Euclid's masterly Book II, on proportion. As Sir Thomas Heath has said, this discovery "must have necessitated a great recasting of the whole fabric of elementary geometry, pending the discovery of the general theory of proportion applicable to incommensurable as well as to commensurable magnitudes." The same writer has also called attention (*The Thirteen Books of Euclid's Elements*, III (2), p. 8) to the interest shown in Book X by the early algebraists, an interest which evidently led, in the 19th century, to a renewed desire to investigate the whole theory of irrational numbers.

It is for these reasons that the publication of the Arabic text and a careful translation of a work hitherto known only imperfectly, is so important. Pappus was not a great mathematician but he was the best of the Greek commentators on Euclid, and the parts of his *Collections* that have come down to us have been carefully studied and highly esteemed by such scholars as Commandino in the 16th century, Wallis in the 17th, Halley in the 18th, and Hultsch and Heiberg in the 19th, with the recent appreciation by Junge and Thomson in the book under review.

The work consists of (1) a series of "Bemerkungen zur dem vorliegenden Kommentar" by Dr. Junge, with notes and a statement of the contents of the commentary; (2) an introduction, describing the manuscript and giving a history of its translation by Abū 'Uthmān al-Dimishqī (10th century), one of the leading physicians of Bagdad and a translator of several important Greek treatises; a discussion of the sources of Pappus's conception of rational quantities; and collation of the Arabic text with the Greek scholia as given in volume V of Heiberg's edition of Euclid; (3) a translation of the text by Mr. Thomson, with numerous notes containing an extensive bibliography; (4) the Arabic text; and (5) a glossary of the technical terms as they appear in the Arabic.

The parts of the work which will be found of greatest value to scholars who do not read Arabic are, of course, the translation and the notes. As to the nature of the former, Mr. Thomson remarks that it "is avowedly of a philological and historical nature and does not pretend to render the thought of Pappus into the terms and signs of modern mathematics." It was therefore desirable that this phase of the mathematical problems should be considered independently, and this has been very acceptably done in the Bemerkungen of Dr. Junge already mentioned. Fortunately, Mr. Thomson had, in his translation, the added assistance of the technical terminology of Sir Thomas Heath's translation of the tenth book of the *Elements*, which he duly acknowledges.

As a whole the book is one of the most encouraging of recent evidences of a new type of scholarship in America and of a desire to clothe the products of such scholarship in a dignified and worthy dress. While we may regret that the Harvard Press was not prepared to print such a treatise in this country, it is to be congratulated that it secured the aid of such a well-equipped organization as the J. J. Augustin Buchdruckerei of Glückstadt and Hamburg with its notable history and its ability to print in no less than twenty-five of the most important of the languages, besides those of Europe and the Americas which use the Roman fonts.

DAVID EUGENE SMITH

YOUNG ON PROJECTIVE GEOMETRY

Projective Geometry. By J. W. Young. Chicago, The Open Court Publishing Co., 1930. ix+185 pp.

This volume is the fourth in the series of Carus Mathematical Monographs sponsored by the Mathematical Association of America. The natural point of departure for the reviewer would seem to be the expressed purpose of the Carus Monographs, namely, "to contribute to the dissemination of mathematical knowledge by making accessible at nominal cost a series of expository presentations of the best thoughts and keenest researches in pure and applied mathematics," "in a manner comprehensible not only to teachers and students specializing in mathematics, but also to scientific workers in other fields, and especially to the wide circle of thoughtful people who, having a moderate acquaintance with elementary mathematics, wish to extend their knowledge without prolonged and critical study of the mathematical journals and treatises." For the present reviewer, this would mean to start with a handicap of skepticism. The "other scientific workers" are finding a steadily increasing need of mathematical training. Their first interests lying in other directions, they naturally wish to secure this training at the least possible cost of time and effort. Each individual will have to find out for himself the minimum expenditure for which he can acquire a useful tool. The Carus Monographs may be helpful to this group. But is it possible to make accessible the keenest researches in mathematics to that wide circle of thoughtful people who wish to extend their knowledge without prolonged and critical study? To ask a simpler question, is it possible to convey to a group of intelligent readers having a moderate acquaintance with elementary mathematics something of the beauty and fascination of the ideas of projective geometry, unless these readers are willing to pay the price of the moderately prolonged and critical study which the average mathematician has devoted to the subject? Whatever the answer to this question, the reviewer ventures the opinion that *if it can be done* Professor Young's book does it.

But if one may be allowed to forget the stated purpose of the Carus Monographs, he is then free to express unreserved approval of this excellent piece of expository writing. The usual first course in projective geometry (so-called) is a mixture of projective and metric geometry in which distance, perpendicularity, and circles are as prominent as harmonic sets, involutions, conics, and polar systems. Probably such courses are useful; but it is doubtful whether they leave the student with any clear conception of what is meant by *projective* geometry. And just at this point one could do the student no greater service than to place in his hands the little book under review.

After defining projective space by introducing the "ideal" elements, the first five chapters proceed to develop pure projective geometry by the synthetic method. The treatment calls upon the reader frequently to exercise his geometric intuitions. It is not at all logical in the sense of building up the subject from an explicitly stated set of postulates; nor is it illogical in any derogatory sense. With respect to this part of the book, one can readily make himself believe in the group of non-mathematical readers, and in their keen interest and pleasure as the subject unfolds itself. Starting with the idea of duality in the plane and in space, one is led on to the consideration successively

of the four-point and four-line in the plane; the complete five-point in space; the Desargues figure as a plane section of the complete five-point in space; harmonic sets of points and lines; projectivities between one-dimensional forms, with the so-called fundamental theorem, and a treatment of double elements; involutions; and conics, with the theorems of Pascal and Brianchon and a treatment of poles and polars. From these eighty pages unmarred by any mention of distance, angle, or circle, a student may catch the real spirit of projective geometry.

In Chapter 6, on Metric Properties, there is clearly set forth the relation between metric and projective properties. It is refreshing to read an exposition in which ordinary metric geometry and projective geometry are *related* without being *confused*. Our school texts in euclidean geometry define parallel lines as lines lying in the same plane but not meeting in a point; teachers who had studied projective geometry began to tell high-school students that these parallel lines "meet in a point at infinity"; and then to avoid the rather bald contradiction they amend the definition of parallels (*in euclidean geometry*) by saying that parallels are lines which do not meet *in any point at a finite distance*! If the teacher of euclidean geometry who has studied projective geometry would read Chapter 6 of this monograph, some of this sort of confusion might be avoided. We find here a clean-cut presentation of the relation between certain ideas of pure projective geometry and such metric concepts as parallel and perpendicular lines; mid-points of line segments and bisectors of angles; types of conics; foci, axes, and directrices of conics; and circles.

Chapter 7 deals with groups of transformations. For one already fairly familiar with the group concept, it is an accurate and lucid presentation. In ten pages one learns of groups, subgroups, transforms of groups, invariant figures under a group, the properties of commutativity and transitivity, and the application of all these ideas to the groups of transformations on one and two-dimensional forms. The reader without considerable mathematical maturity will doubtless find this rather hard going. In discussing the types of point-transformations on a plane, the case with one real and two imaginary fixed points seems to have been left out. One might suppose that it was the intention to include it in Type I if it were not for the statement that "the projectivities on the sides of the invariant triangle are all hyperbolic."

In Chapter 8 there is an introduction of analytic methods based on an algebra of points which is itself based on the group concepts of the preceding chapter. An important fact is given emphasis at this point, namely, that a coordinate system suitable for the study of projective properties may be set up without any use of distance or other metric ideas. The ninth and last chapter presents Klein's concept of a geometry associated with or defined by a group of transformations; and treats very briefly the affine, euclidean metric, and non-euclidean geometries from this point of view.

In the opinion of the reviewer, the book meets a real need in the literature of mathematics in English, and the author has performed a real service in the writing of it.

The Open Court Publishing Company has brought out the book in the very satisfactory size, binding, and typography uniform with the preceding volumes of the series.

W. B. CARVER

DUNHAM JACKSON ON APPROXIMATION

The Theory of Approximation. By Dunham Jackson. New York (American Mathematical Society Colloquium Publications, Volume 11), published by this Society, 1930. v+178 pp.

In 1885 Weierstrass announced his now famous theorem: *Any continuous function $f(x)$, defined over a finite interval (a, b), can be approximated in (a, b) uniformly and indefinitely by a sequence of polynomials $P_n(x)$, whose degrees n tend to infinity*, that is, $\lim_{n\to\infty} P_n(x)-f(x)=0$, $(a\leqq x\leqq b)$. The same holds true, if polynomials are replaced by finite trigonometric sums of ever increasing orders. We shall speak of this, for brevity, as the (ordinary) polynomial or trigonometric approximation.

One can hardly overestimate the influence exerted by this theorem, now commonly known as "Weierstrass' Theorem," on the development of modern analysis. It suffices to say that the theorem in question very often enables the investigator to extend, at a single stroke, a property previously established *for polynomials only* to the infinitely wider class of continuous functions.

It seems, therefore, quite fitting that the *Theory of Approximation* is the subject of one of the series of the Colloquium Lectures of the Society, a subject for which no one is better qualified as lecturer than Professor Dunham Jackson, who has made to it such important contributions. Those who had the pleasure of hearing the Colloquium Lectures or other papers read before the Society by Professor Jackson always admire their lucidity and elegance of style. The present book possesses these qualities to a high degree. The exposition is clear, detailed and interesting, and in many points distinctly novel.

The book consists of five chapters. Chapter 1 is devoted to the ordinary trigonometric and polynomial approximation of continuous functions. After a brief introduction, where Weierstrass' Theorem is stated, the author proves the fundamental Theorem 1 on trigonometric approximation of a function $f(x)$ satisfying a Lipschitz condition. The proof is based upon the use of the *singular* integral

$$h_m \int_{-\pi/2}^{\pi/2} f(x+2u)F_m(u)du, \quad F_m(u)=\left(\frac{\sin mu}{m\sin u}\right)^4, \quad \frac{1}{h_m}=\int_{-\pi/2}^{\pi/2} F_m(u)du,$$

introduced by Jackson in his Thesis; the analysis is carried out in a detailed manner. This leads to Theorem 2 on trigonometric approximation of any continuous function, given its modulus of continuity $\omega(\delta)$, through an ingenious device also introduced by the author in his Thesis. A closer analysis of the approximating sum employed in Theorem 1 leads further to Theorems 3, 4 on trigonometric approximation of a function whose pth derivative satisfies a Lipschitz condition or has a given modulus of continuity. In the next section the author states similar theorems for polynomial approximation, reducing this to the previous case by means of the substitution $x=\cos\theta$. The next section is devoted to a brief discussion of the rapidity of convergence of Fourier and Legendre series, which is taken up with more details in Chapter 2. We note an extremely elegant proof (pages 27–28) of the following property of Legendre's

polynomials $X_k(x)$ (of degree k): $|X_k(x)| \leqq c/(k^{1/2}(1-x^2)^{1/2})$, $(-1<x<1$; $c=$const.). The chapter closes with a brief remark on approximation by means of more general classes of functions.

Chapter 2 deals with the approximation of discontinuous functions, especially of those of bounded variation. The central theme is Fourier series. The author uses the Lebesgue theory of integration and discusses the convergence of Fourier series, first, for summable functions satisfying in a part of the period certain conditions of continuity. Due to the general character of the functions under consideration, the degree of convergence is not discussed, but we find a clear and interesting discussion of it in the following section dealing with functions of bounded variation or whose derivative of a certain order is of bounded variation. Next comes the convergence and the degree of convergence of the first arithmetic mean, *Fejér sum*, of the Fourier series for a given summable function, and its degree of convergence for functions satisfying certain conditions of continuity. Here (page 63) the author makes the interesting remark that no higher order of approximation of $f(x)$ by means of its Fejér sum would be obtained by supposing $f(x)$ provided with additional derivatives (besides $f'(x)$), using as illustration the analytic function $\cos x$. A similar discussion is further carried out for Legendre series. Here (page 73) we notice again an elegant estimate of $|X_{n-1}(x)-X_{n+1}(x)|$ by means of the Laplace integral used in Chapter 1.

Chapter 3 is devoted to trigonometric and polynomial approximation based on the principle of least squares or, more generally, of least mth powers, that is, the approximation of $f(x)$ by $\phi(x)$ on (a, b) being measured by

$$\int_a^b \rho(x)\,|f(x)-\phi(x)|^m dx, \qquad (m>0; \quad \text{weight-function } \rho(x) \geqq 0 \text{ on } (a,b)).$$

The minimum property involved in the least squares principle is established, first, for any set $p_0(x), p_1(x), \cdots$ of normalized orthogonal functions over a given interval (a, b), and later applied to the set of trigonometric functions $\{\sin mx, \cos mx\}$, $(m=0, 1, \cdots)$, where the least squares principle leads to the Fourier expansion of the function we seek to approximate. Sufficient conditions for the convergence of the minimizing sum to $f(x)$, under proper conditions of continuity, are then established in a simple and elegant manner, by means of the well known Bernstein Theorem: $\max |T_n(x)| \leqq L$ implies $\max |T_n'(x)| \leqq nL$, ($T_n(x)$ a trigonometric sum of order n). There follow various generalizations: introduction of a weight-function, together with the least mth powers. The results obtained are then extended to the case of polynomial approximation, by making use of the generalized Bernstein Theorem and its various corollaries as applied to polynomials. Especially noteworthy is Corollary III (page 94):

$$|P_n(x_2)-P_n(x_1)| \leqq \frac{4nL}{(b-a)^{1/2}}|x_2-x_1|^{1/2}, \text{ if } |P_n(x)| \leqq L \text{ on } (a, b),$$

$$(a \leqq x_1, x_2 \leqq b;\ P_n(x) \text{ a polynomial of degree } n).$$

Legendre's series here again presents itself as the simplest case ($\rho(x)\equiv 1$, $m=2$). The chapter ends with the least squares polynomial approximation and its convergence over the infinite interval $(-\infty, \infty)$, with the weight-function e^{-x^2}, that is, using Tchebycheff-Hermite polynomials. The author justly re-

marks in closing that the method employed yields little, if anything new, being of an elementary nature (it does not make use, for instance, of the asymptotic expression for a Tchebycheff-Hermite polynomial of a very high degree).

Chapter 4 is devoted to trigonometric interpolation, intending to show, as the author says at the beginning, "certain striking analogies $\cdots$ between the theory of interpolation by means of trigonometric sums and that of Fourier series." An interpolation formula is given with equidistant abscissas, which, regarding its coefficients and convergence properties, bears a close resemblance to the partial sum of the Fourier expansion. The degree of its convergence is discussed for functions continuous over the entire period or over a part of it, or of bounded variation. The discussion follows the same lines as used previously (Chapters 1 and 2), and the results are very similar, thus confirming the passage quoted above. In the second part of this chapter the author derives an interpolation formula (introduced by himself in analysis) which possesses the remarkable property that *it converges uniformly for any continuous periodic function* (to the value of the function in question), in common with the Fejér sum discussed in Chapter 2. Its degree of convergence is discussed for functions satisfying a Lipschitz condition.

Chapter 5 deals with the fundamental notions of geometry of function space. We find here, first, the least squares approximation principle applied to "polynomials" $\sum_{k=0}c_k p_k(x)$, (c_k=const.), $\{p_k(x)\}$ representing a general set of linearly independent functions defined over (a, b). The author proceeds with the geometric interpretation in function space of distance, orthogonality, and, more generally, of the notion of an angle. This yields an interesting interpretation of coefficients of correlation. The chapter ends with geometrical considerations briefly applied to frequency functions and to vector analysis in function space.

In any given book one can find shortcomings and omissions which very often represent a function of the mental and artistic tastes of the reviewer Such, perhaps, is the case with the remarks which follow.

We find missing some commonly used terms which, in some instances, could have shortened subsequent statements. Such is the case with the term "Lipschitz condition": $|f(x_2)-f(x_1)|\leqq\lambda|x_2-x_1|$, introduced on page 2 and used extensively throughout the book (later, page 22, the author uses the term "Lipschitz-Dini condition" for $\lim_{\delta\to 0}\omega(\delta)\log\delta=0$). The same must be said of the term "Fejér sum" for the first arithmetic mean of partial sums of Fourier series (page 57) (later the author refers to it as "Fejér mean"), of the term "Laplace integral" for the integral $(1/\pi)\int_0^\pi[x+i(1-x^2)^{1/2}\cos\phi]^k dx$ (page 26), also of the term "Schmidt's orthogonalization process" (page 90).

Some proofs could be abbreviated, as, for example, that on page 27, or on pages 78–79, where the author proves that, given a sequence $P_0(x), P_1(x), \cdots$ of normalized orthogonal functions, summable with their squares on (a, b), $\min\int_a^b[f(x)-\sum_{k=0}^n c_k P_k(x)]^2dx$ is attained, if and only if $c_k=\int_a^b f(x)P_k(x)dx$. It suffices to consider, we believe, the expression

$$\int_a^b\Big[f(x)-\sum_{k=0}^n(h_k+c_k)P_k(x)\Big]^2dx$$

$$=\int_a^b f^2(x)dx-2\sum_{k=0}^n(c_k+h_k)c_k+\sum_{k=0}^n(c_k+h_k)^2$$

$$= \int_a^b f^2(x)dx - \sum_{k=0}^{n} c_k^2 + \sum_{k=0}^{n} h_k^2$$

$$\geqq \int_a^b f^2(x)dx - \sum_{k=0}^{n} c_k^2,$$

where c_k is as stated above, and h_k is an arbitrary constant; and the statement given above follows directly.

On the other hand, on page 69 it seems advisable to state *fully* Schwartz's inequality, which still has not (although it should have) found its way into our textbooks. Also the statement concerning the limit of the approximation of an arbitrary continuous function (page 8) should be stated more clearly (as in the author's thesis): $\psi(n)$ being an arbitrary positive non-increasing function such that $\lim_{n\to\infty}\psi(n)=0$, there exists a function $f(x)$, continuous in (a, b), for which $\psi_f(n)\geqq\psi(n)$, $\psi_f(n)$ = the best approximation of $f(x)$ in (a, b) by means of a polynomial of degree n. Also the process of orthogonalizing a given sequence of functions $\{(q_n(x)\}$ (page 90) can be made clearer and more easily to be memorized by the simple remark that each new orthogonalized $Q_k(x)$ is the remainder in the formal development of $q_k(x)$ in a series of previously orthogonalized $Q_0(x), Q_1(x), \cdots, Q_{k-1}(x)$.

May we add a few more remarks. At the beginning the reviewer would like to see introduced, if briefly, the notion of "best approximation," in the Tchebycheff sense. It plays such an important role in the theory of approximation and could be used also, when discussing the degree of convergence of Fourier and Legendre series, in order to give the reader an idea how favorably the approximation in question compares with the best possible one.

When introducing the integral $h_m\int_{-\pi/2}^{\pi/2} f(x+2u)\,[\sin mu/(m\sin u)]^4 du$ (page 3), one would like to see a reference to *singular integrals* in general, stating the characteristic properties of the integrand to which the success of their application is due. Similarly, when discussing Legendre series, one might mention, in connection with the order of magnitude of $\int_{-1}^{1}|\sum_n(x, t)|\,dt$ (pp. 31–32), the so-called "Lebesgue constants" and their role in the theory of series of orthogonal functions in general. Furthermore, the close similarity of the results obtained for Legendre series to those established for Fourier series makes very desirable a remark to the effect that this similarity is essentially due to the fact that for $-1<x<1$ the normalized Legendre polynomial $(2k+1/2)^{1/2}X_k$ behaves, for k very large, like $\cos kx$ or $\sin kx$. (In fact, we know at present how closely related are, with regard to convergence and divergence, these two series.)

When dealing with approximation based on the least mth powers principle, one wishes to see mentioned the result of Pólya (Comptes Rendus, vol. 157 (1913)) establishing a remarkable relation between this method of approximation and the (best) ordinary one.

Finally, when discussing the trigonometric interpolation formula analogous to Fejér's mean (pp. 142–148), the author makes a parenthetical reference to an article by Fejér. We believe this reference should be amplified by citing from that article Fejér's interpolation formula, which has the same important property of being convergent for any continuous function.

The reviewer believes that remarks of the kind mentioned above, even

made in footnote form, tend to enlarge the horizon of the reader and to stimulate his mental curiosity.

The aforesaid critical remarks do not, of course, detract from the merit of the book under review. The reader will enjoy it as interesting, stimulating, extremely well written, and beautifully printed. Many a worker will be inspired by the pages devoted to Fourier and Legendre series and to Tchebycheff-Hermite polynomials to try his hand in obtaining similar results for other classes of orthogonal functions, and in particular, for other classes of orthogonal Tchebycheff polynomials.

As to the choice of material, the reviewer would prefer to see the last chapter devoted not to geometry of function space, but to approximation over the real axis of functions with singularities. However, in the preface the author expresses himself as follows: "The title of this volume is an abbreviation for the more properly descriptive one: *Topics in the Theory of Approximation*. It is an account of certain aspects and ramifications of a problem to which I was introduced at an early stage, and which has given direction to my reading and study ever since." This is very disarming indeed. The author is certainly the supreme judge in choosing his material.

May we, therefore, express the hope which, we feel certain, is shared by all readers of this book, that it will be soon followed by a second one devoted to the same important field, written, needless to say, with the same mastery of style and scholarly authority.

J. SHOHAT

SHORTER NOTICES

Atoms, Molecules and Quanta. By Arthur Edward Ruark and Harold Clayton Urey. New York, McGraw-Hill Book Co., International Series in Physics, 1930. xvii+790 pp.

This substantial volume is an outstanding contribution to the rapidly growing list of books on the quantum theory of atomic structure. Its value to the student consists not only in its vast scope—it far outreaches any recent text in the English language in this respect—but mainly in the method of attack, the reasons for which are clearly set forth in the preface and with which the reviewer is in entire sympathy. From the pedagogical point of view it is very clear that the only understandable way to proceed to the newer theories of atomic structure is through the Bohr theory. In spite of its difficulties the latter presents a vivid picture which can be grasped by the student taking up the subject for the first time. Moreover many will be found to agree heartily with the authors' opinion that the "modelmässig" mode of approach to the understanding of atomic structure will for some time to come continue to be of fundamental importance to workers in physical science. This does not at all imply a lack of sympathy with the newer program with its more highly abstract postulational treatment. Physical theories will continue to be judged on the fundamental bases of simplicity of structure and the utility and verifiability of the predictions implied in them. But there are psychological factors involved and it seems that physicists will not soon forego the use of pictures in the construction of theories.

For these reasons it is particularly refreshing to find in the present work a thorough survey of the Bohr theory and the problems which led to its formulation as well as a good introduction to wave mechanics and matrix mechanics. The experimental physicist will welcome the careful discussion of fundamental experiments, and the correlation of a wealth of data hardly available elsewhere in a single volume. The theoretical physicist will find a satisfactory review of advanced dynamics adequate for the understanding of the mechanics of multiply periodic systems. A particularly strong feature of the work in the eyes of many physicists will be the chapters on spectroscopy, embracing almost one half of the whole volume, and covering very fully optical line spectra, molecular spectra and X-ray spectra. The authors here have not hesitated to draw heavily on the vector model for their discussion, but have also emphasized the increasing contributions being made in this field by the applications of the new quantum mechanics.

The formal treatment of the new quantum mechanics which occupies the last two hundred pages of the book is clear and logically presented. The wave mechanics is first introduced through de Broglie's theory of matter waves, and the transition is then made to the Schrödinger assumption in pointing out Hamilton's original treatment of the similarity between geometrical optics and the dynamics of a particle. The "eigenwert" problem is treated in a way which, while not entirely rigorous, will be clear to the average student. More com-

plete mathematical references are given in an appendix. The same is true of a number of other purely mathematical topics, such as complex integration, orthogonal functions, curvilinear coordinates, etc. A whole chapter is devoted to the study of hydrogen atoms by the wave mechanics method with sufficient concrete details and applications to make clear to the student the utility of the method.

The matrix mechanics is then introduced and developed as an independent theory, with detailed study of applications. This is followed by a discussion of the connection between the wave and matrix mechanics, and the method of constructing the quantum matrices from the solutions of the wave equation. In the chapter on the general theory of quantum dynamics, there is a thorough treatment of Heisenberg's indetermination principle, and the transformation theory of Jordan and Dirac. The frequent introduction of concrete illustrations greatly enhances the value of this chapter.

The book is concluded with chapters on the treatment of non-hydrogenic atoms and molecules by the new mechanics, spectral intensities and the diffraction of electrons and atoms by crystals.

The style of the book is in general clear and concise. The typography is excellent and the text is well illustrated by a large number of well-made diagrams. On the whole it is a work which may be heartily recommended to all those interested in the problems of atomic structure.

R. B. LINDSAY

Leçons sur les Ensembles Analytiques et leurs Applications. By Nicolas Lusin. With a preface by Henri Lebesgue and a note by Waclaw Sierpinski. Paris, Gauthier-Villars, 1930. xvi+328 pages.

This volume in the Borel series contains a systematic survey of the present knowledge of analytic sets, a knowledge which is chiefly due to the researches of the Russian mathematician who is the author of this book. In fact the only results which are not due to Lusin or his pupils come from members of the Polish school of Sierpinski and Mazurkiewicz. The analytic sets of Lusin, which are a generalization of Borel sets, have been briefly mentioned previously in several books (Hausdorff's *Mengenlehre*, for instance), but this is the first book devoted entirely to their study.

Lebesgue in his preface humorously points out that the origin of the problems considered by Lusin lies in an error made by Lebesgue himself in his 1905 memoir on functions representable analytically. Lebesgue stated there that the projection of a Borel set is always a Borel set. Lusin and his colleague Souslin constructed an example showing that this statement was false, thus discovering a new domain of point sets, a domain which includes as a proper part the domain of Borel sets. Lebesgue expresses his joy that he was inspired to commit such a fruitful error.

Of the five chapters of approximately equal length into which the book is divided, the first two are devoted to Borel sets. Here and throughout the book, Lusin considers as his fundamental domain the set of irrational points of a linear space. By excluding the rational points, certain simplifications in statements and proofs of theorems are obtained. After mentioning several different methods of defining Borel sets and showing their logical equivalence, a study is

made of the structure of Borel sets from a geometric standpoint. Theorems are obtained describing the structure of a point set of a given class of Baire-de la Vallée Poussin. In particular, sets of class 0 and class 1 are described and the constructive existence of sets of classes 1, 2, 3, and 4 is shown. By showing the existence of universal elements in space of two dimensions, that is two-dimensional sets of class α such that any linear set of class α is obtained by cutting the set by a properly chosen line, the effective existence of linear sets of each class is shown.

After this introduction, the author shows in Chapter 3 how the concept of a Borel set may be generalized to that of an analytic set, and the properties of analytic sets are then studied in some detail. In this connection Lusin makes much use of the ingenious device which he calls a sieve (*crible*). The original notion of this device Lusin modestly attributes to Lebesgue, while the latter generously states that Lusin has read into Lebesgue's paper ideas which were not there but which are due entirely to Lusin himself. Be that as it may, Lusin shows by means of a sieve that Lebesgue has given in his 1905 memoir the first example of an analytic set not a Borel set that exists in mathematical literature. The reason for the friendly disagreement mentioned above is that there is a difference of view-point involved. Lebesgue was interested in analysis only, and hence did not attempt a geometric interpretation of the function which he had obtained in his memoir. Lusin, on the other hand, was interested in Lebesgue's example from a geometric standpoint and his geometric interpretation of it brings to light certain implications of which Lebesgue was not aware. Except for this example due to Lebesgue, the results of Chapter 3 are almost exclusively the work of Lusin and Souslin.

Then follows a brief fourth chapter on implicit functions, constituting the "applications" mentioned in the title of the book. Lusin shows how the problem of the domain of existence of a function of Baire is tied up with the geometric properties of that domain from the standpoint of analytic sets.

The final chapter is composed of two distinct parts, a study of projective sets and an analysis of the memoir of Lebesgue which has been referred to previously. In the analysis of Lebesgue's memoir, its connections with the present book are pointed out and Lusin's point-set point of view serves to complement Lebesgue's functional point of view.

The class of all projective sets forms a still more general class than that of the class of all analytic sets, and includes as well the class of all sets complementary to analytic sets. In this part of the book the results are due to Mazurkiewicz and Sierpinski rather than to Lusin, although the latter has contributed to this topic also. The book concludes with a two-page note by Sierpinski on a topic connected with projective sets.

The book as a whole is an admirable presentation of a subject in which research is now being actively pursued, and should prove invaluable to newcomers in the field. The author gives copious references to the literature for those who wish to pursue further the topics discussed. He also takes pains to point out the outstanding unsolved problems in the field, their implications, and their connections with present scientific knowledge in this domain of mathematics.

H. M. GEHMAN

Mathematische Formelsammlung. By O. Th. Bürklen, revised by F. Ringleb. Berlin and Leipzig, de Gruyter, 1931. 255 pp.

This volume is Number 51 of the Sammlung Göschen, the first edition of which appeared in 1927 and was reviewed in this Bulletin, vol. 34, page 669. In the new edition there has been no change in the general plan of the book. The astronomical data in Chapter IX have been corrected and a few additions have been made to the chapters on analytic geometry and calculus. One chapter has been added and gives the fundamental formulas of vector analysis.

W. R. LONGLEY

Theoretical Mechanics. By William Duncan MacMillan. *Statics and Dynamics of a Particle.* pp. xviii+430. 1927. *The Theory of the Potential.* pp. xii+469. 1930. New York, McGraw-Hill.

The first volume of this series by Professor MacMillan deals with the foundations of mechanics, the statics of particles, rigid and deformable bodies including elastic solids, and the dynamics of a single particle. A later volume covering the omitted phases of dynamics is promised. The book is clearly written, and is a suitable text for students with a good grounding in the calculus, introducing them to the equations of Lagrange and Hamilton. The main treatment of dynamics is based on Newton's laws, with clear indications as to the definitional character of parts of these. Later, alternative treatents are indicated.

A few statements in the text require amplification. For instance, on page 2, the statement that vectors can be moved about freely is followed by the remark that forces are typical vectors. Again on page 133, a definition of number of degrees of freedom is found which would not apply to non-holonomic systems.

In the second volume, there is collected a large amount of material on the theory of the potential function. The emphasis is on methods capable of concrete physical application, rather than on those of the greatest generality. However, the standard of rigor is relatively high, and while practically no results of the past twenty-five years are expounded in the text, the bibliography includes references to recent summaries of modern work. Although the presentation is clear in detail, there is a lack of indication of the relation of the parts of the subject to one another and to other fields of mathematics. For example, the Dirichlet principle is explained, with the Weierstrass critique, but without an explicit indication to the student that two methods of rigorously establishing the existence in question are to be given in the next chapter. Similarly, the analogy of the expansion of a function in a series of spherical harmonics to the more familiar Fourier expansion is not mentioned. In the proof of this expansion theorem (p. 386) the author writes "generally continuous" function, when he means *differentiable* function, as otherwise the derivatives which appear on integrating by parts need not exist. This misuse of terms was a little more pardonable in Darboux's paper of 1874, which the author follows, than in a book of 1930. This volume on potential theory may serve those who wish a revised text along the lines of B. O. Pierce's *Newtonian Potential Function.*

PHILIP FRANKLIN

Introduction to the Theory of Fourier's Series and Integrals. By H. S. Carslaw. Third edition, revised and enlarged. London, Macmillan and Company, 1930. xiii+368 pp.

Fouriersche Reihen. By Werner Rogosinski. (Sammlung Göschen, No. 1022.) Berlin and Leipzig, de Gruyter, 1930. 135 pp.

The second edition of Professor Carslaw's book was reviewed in vol. 28 (1922) of this Bulletin. The third edition, while substantially of the same scope as the second, is in no sense a mere reprinting of the latter edition. There has been considerable rewriting throughout the book and a number of noteworthy additions. The revisions are all advantageous and serve to bring the work into closer harmony with the present status of the theory of Fourier series.

The most important addition is an appendix on Lebesgue's theory of integration, with a few of the applications of this theory to the discussion of Fourier series. In a combined review of the second edition of Carslaw's book (see reference above) and volume I of the second edition of Hobson's *The Theory of Functions of a Real Variable and the Theory of Fourier's Series* the present reviewer expressed the opinion that sooner or later certain workers in the field of applied mathematics might well find their center of interest transferred from the Riemann integral to the Lebesgue integral. The appearance of a treatment of the Lebesgue theory in a book written primarily for those interested in the applications of Fourier series seems to prove that the drift of the times is in line with the reviewer's prediction. There is a certain neatness and elegance in the theory of Fourier series, as based on the Lebesgue definition of the integral, which is lacking in any discussion that is limited to the Riemann definition. This leads to the inevitable conclusion that the Lebesgue integral is the natural tool in dealing with Fourier series and that it must eventually be introduced even in books on the applications. It seems to the reviewer that it is only a question of time until the elements of the Lebesgue theory filter down into such courses as advanced calculus, and then there will be no valid reason for avoiding it in books on Fourier series.

The treatment of the Lebesgue integral in Appendix II of Carslaw's book is an excellent brief discussion along the lines of the expository treatments due to de la Vallée Poussin and is a valuable addition to the available literature on this fundamental topic. The other additions to the book that should be mentioned are: an extension and revision of the historical introduction to take fuller account of recent literature; more complete discussion of monotonic sequences, upper and lower limits of indetermination, second law of the mean, uniform convergence, and term by term integration; and finally the inclusion of the Heine-Borel theorem, Parseval's theorem, and a discussion of functions of bounded variation.

Professor Rogosinski's book deals exclusively with the theory of Fourier series and presupposes familiarity with various topics from the general theory of functions of a real variable treated by Carslaw. Therefore, although the general policy with regard to the series in which the book appears has restricted the author to a volume of much smaller extent, he has succeeded in discussing virtually all the topics directly connected with Fourier series that are treated in Carslaw's book, though not always in as much detail. In addition he touches on certain additional topics, such as the uniqueness of the development. Like

Carslaw he bases his treatment of the subject on the Riemann integral, though referring occasionally to generalizations that only have significance in the domain of the Lebesgue theory.

Between the table of contents and the text there is given a brief list of books that go more deeply into the theory of Fourier series than the present work. This list furnishes a curious instance of the failure of some continental writers to keep up to date with regard to works published in English. It contains a reference to the first edition of Hobson's *The Theory of Functions of a Real Variable and the Theory of Fourier's Series* (which dates back to 1907), but makes no mention of the second edition in two volumes (vol. I (1921), vol. II (1926)), or the third edition of volume I (1927), although continental works of later date are cited. In view of the fact that volume II of the second edition of Hobson's work contains the most extensive account of existing literature on Fourier series available in any language, such an omission is difficult to explain.

C. N. Moore

Geometrische Transformationen. By Karl Doehlemann. Second edition prepare., by Wilhelm Olbrich. (Göschens Lehrbucherei.) Berlin, de Gruyter and Cod 1930. vi+254 pp.

In this one volume Professor Olbrich presents a revision of the two-volume work of Doehlemann, published in 1902. To condense the material into one volume, some topics in projective transformations have been omitted, as, for example, the linear transformations of a quadric surface into itself. The material of the second volume of the old edition, which was devoted to quadratic and higher birational transformations, has been cut down greatly. The new edition contains a brief exposition only of quadratic transformations in the plane and in space, inversion in the plane and in space, and simple transformations in the complex plane.

P. F. Smith

NOTES

A new monthly abstract journal for mathematics and the allied sciences has been established, under the editorship of O. Neugebauer; it is entitled Zentralblatt für Mathematik und ihre Grenzgebiete (reine und angewandte Mathematik, theoretische Physik, Astrophysik, Geophysik). The board of editors is as follows: P. Alexandroff, J. Bartels, W. Blaschke, R. Courant, H. Hahn, G. H. Hardy, F. Hund, G. Julia, O. D. Kellogg, H. Kienle, T. Levi-Civita, R. Nevanlinna, H. Thirring, B. L. van der Waerden. It is published by Julius Springer; price 48 rentenmarks per annual volume.

Publication of a new series (series 2) of Annali della R. Scuola Normale Superiore di Pisa will begin in November, 1931. The publication will be in two separate sections, one in mathematics and mathematical physics and one in literature. The section in mathematics and mathematical physics will be edited by the science faculty of the University of Pisa under the direction of Professor Leonida Tonelli, Director of the Mathematical Institute of the University and Professor at the Scuola Normale. One volume of this section will be published annually, consisting of four parts of about a hundred pages each. It will be published by Nicola Zanichelli, Bologna; price 100 lire per annual volume.

A new journal devoted to mathematical recreations, entitled Revue Périodique des Questions Récréatives, has been established under the editorship of M. Kraïtchik, of Brussels.

The firm of G. E. Stechert and Company, of New York, announce that they have reprinted *The Introduction to the Lie Theory of One-Parameter Groups*, by A. Cohen, long out of print and scarce. They have also in preparation reprints of Boole's *Differential Equations* and Bolza's *Calculus of Variations*.

The next International Mathematical Congress will be held in September, 1932, at Zurich, Switzerland.

According to a notice issued, in Latin, by the Pontifical Academy of Sciences (Nuovi Lincei), that Academy is inaugurating the broadcasting of notes and information on mathematics and the experimental sciences and their applications. In general, the language used will be Latin, but for notes referring to investigations or discoveries the author's own language may be used. A general invitation to supply material for this purpose is expressed, and all communications submitted will be considered by a committee of the Academy.

A symposium on *Mathematics in the service of chemistry* was held by the American Chemical Society at its spring meeting at Indianapolis. The addresses presented at this symposium have been published in the Journal of Chemical Education for June, 1931, and include articles on thermodynamic calculations, quantum theory, and the structure of the atom with particular reference to valence.

The Royal Belgian Academy announces the following subject for its prize in mathematics to be awarded in 1932: *to complete, in some important respect, recent researches on the properties of functions univalent in a given area.* This Academy has awarded its F. Deruyts prize to Professor R. Deaux, of the School of Mines at Mons, for his work in synthetic geometry.

Professor Ganesh Prasad, of the University of Calcutta, has presented the sum of 1,400 rupees to the Calcutta Mathematical Society to found a prize and medal in memory of his daughter.

President K. T. Compton, of the Massachusetts Institute of Technology, has been awarded the Rumford medal of the American Academy of Arts and Sciences, for his researches in thermionics and spectroscopy.

Dr. Werner Heisenberg, professor of theoretical physics at the University of Leipzig, has been awarded the Barnard medal of Columbia University.

The Daniel Guggenheim gold medal for notable achievement in aeronautics has been awarded to Dr. F. W. Lancaster, of Birmingham, England, for his contributions to the fundamental theory of aerodynamics.

Professor Emile Cotton has been elected correspondent of the Paris Academy of Sciences, in the section of geometry.

Professor Albert Einstein has received the honorary degree of doctor of science from the University of Oxford.

Professor Maurice Fréchet has been elected foreign member of the Polish Academy of Sciences.

Dr. Ernst Lindelöf, of Helsingfors, has been elected corresponding member of the Prussian Academy of Sciences.

Professor Gino Loria, of the University of Genoa, has been elected member of the Masaryk Academy of Prague.

Sir William Bragg has been awarded the Order of Merit.

Sir Arthur Eddington has been elected president of the London Physical Society.

Professor Horace Lamb, of the University of Manchester, has been knighted.

The University of St. Andrews has conferred honorary doctorates on Sir J. H. Jeans and the Reverend Alfred Young.

Lord Rutherford has been elected president of the British Institute of Physics.

President K. T. Compton, of the Massachusetts Institute of Technology, has received an honorary doctorate of science from Stevens Institute.

Professor P. Debye, of the University of Leipzig, has been elected foreign associate of the National Academy of Sciences.

Sir Arthur Eddington, of Cambridge, has been elected foreign member of the American Philosophical Society.

Professor L. P. Eisenhart, of Princeton, has received the honorary degree of doctor of science from Columbia University.

Sir J. H. Jeans has received an honorary doctorate of laws from Johns Hopkins University.

Professor R. L. Moore, of the University of Texas, has been elected member of the National Academy of Sciences.

Professor George Rutledge, of the Massachusetts Institute of Technology, has been elected fellow of the American Academy of Arts and Sciences.

In addition to his lectures at the Collège de France, mentioned in the May issue of this Bulletin, Professor George David Birkhoff, of Harvard University, lectured on May 1 at the Institut Henri Poincaré on *Quelques questions sur les équations différentielles dans le domaine réel*, on June 3 before the Société Mathématique de France on *Quelques courbes fermées remarquables*, and on June 5 before the Société Philosophique de France on *Une théorie quantitative de l'esthétique*. His mathematical lectures will appear in the Journal de Mathématiques, in the Bulletin de la Société Mathématique, and in the Annales de l'Institut Henri Poincaré.

Professor Wilhelm Blaschke, visiting lecturer of the American Mathematical Society, delivered the following lectures in the spring of 1931: at Lehigh University, April 10, *On foundations of geometry*; at the University of Pennsylvania, April 24, *Topological questions of differential geometry*; at George Washington University, May 16, *Foundations of differential geometry*.

Professor Karl Federhofer, of Graz, has been appointed professor of technical mechanics at the Munich Technical School.

Dr. T. H. Kiang has been appointed professor of mathematics at the Peking National University.

Professor Jacques Chapelon has been appointed to the professorship of rational mechanics and the calculus of probabilities at the University of Lille.

Professor J. Kampé de Fériet, "professor sans chaire" at the University of Lille, has been appointed to the professorship of the mechanics of fluids.

Professor Sebastian Finsterwalder, of the Munich Technical School, has retired.

Dr. Viktor Kommerell has been appointed honorary professor of mathematics at the University of Tübingen.

Dr. H. Hopf has been appointed professor of mathematics at the Zurich Technical School.

Professor E. W. Hobson, of the University of Cambridge, will retire in September, 1931.

Dr. R. C. J. Howland has been appointed reader in mathematics at University College, London.

Assistant Professor Evelyn T. Carroll has been promoted to an associate professorship of mathematics at Wells College.

Dr. L. W. Cohen has been appointed assistant professor of mathematics at the University of Kentucky.

Assistant Professor T. F. Cope has been promoted to a professorship of mathematics at Marietta College.

Dr. F. W. Doermann, of New York University, has been promoted to an assistant professorship of physics.

Associate Professor H. H. Downing, of the University of Kentucky, has been promoted to a professorship of mathematics.

Dr. W. L. Duren, of the College of the City of Detroit, has been appointed assistant professor of mathematics at Tulane University.

Dr. W. W. Flexner has been appointed lecturer at Bryn Mawr College.

Associate Professor M. C. Foster has been promoted to a professorship of mathematics at Wesleyan University.

Dr. H. Hencky, of the Delft Technical School, has been appointed associate professor of mechanics at the Massachusetts Institute of Technology.

Dr. Deborah M. Hickey has been appointed professor of mathematics at the Mississippi Delta State Teachers College.

Mr. E. H. Hildebrandt has been appointed assistant professor of mathematics at De Pauw University.

Mr. A. S. Householder has been promoted to an assistant professorship of mathematics at Washburn College.

Assistant Professor W. G. Hubert has been promoted to an associate professorship at the College of the City of New York.

Dr. H. K. Hughes has been appointed assistant professor of mathematics at Purdue University.

Professor Alfred Hume, of Southwestern University, Memphis, has been appointed president of Branham and Hughes Military Academy, Spring Hill, Tennessee.

Professor J. A. Hurry, of Western State College, Gunnison, Colorado, has been appointed head of the department of physics at San Antonio Junior College.

Assistant Professor F. E. Johnston, of George Washington University, has been promoted to an associate professorship of mathematics.

Associate Professor W. C. Krathwohl, of the Armour Institute of Technology, has been promoted to a professorship of mathematics.

Dr. K. Lanczos, professor of mathematics and mathematical physics at the University of Frankfurt, has been appointed visiting professor of mathematics at Purdue University for the year 1931–32.

Professor Alfred Landé, of the University of Tübingen, has been appointed professor of theoretical physics at Ohio State University.

Associate Professor H. F. MacNeish, of the College of the City of New York, has been transferred to Brooklyn College.

Dr. J. E. Merrill has been appointed assistant professor of astronomy at the University of Illinois.

Assistant Professor W. M. Miller, of Marquette University, has been appointed to an assistant professorship at Tufts College.

Dr. A. K. Mitchell, of Yale University, has been appointed to an assistant professorship at Trinity College, Hartford.

Professor U. G. Mitchell, of the University of Kansas, has been appointed head of the department of mathematics, succeeding Professor C. H. Ashton, who will, however, continue his teaching schedules.

Dr. D. C. Morrow has been promoted to an assistant professorship at the College of the City of Detroit.

Associate Professor D. S. Morse has been promoted to a professorship of mathematics at Union College.

Dr. C. V. Newsom has been appointed associate professor of mathematics at the University of New Mexico.

Professor E. J. Oglesby, of Washington Square College, New York University, has been appointed professor of engineering mathematics at the University of Virginia.

Dr. Anna Pell-Wheeler will resume her position as head of the department of mathematics at Bryn Mawr College in the fall of 1931.

Assistant Professor G. A. Pfeiffer, of Columbia University, has been promoted to an associate professorship of mathematics.

Professor C. L. Poor, of the department of celestial mechanics at Columbia University, has retired.

Miss Mary J. Quigley, of Boston Teachers College, has been promoted to a professorship of mathematics.

Professor G. E. Raynor of the University of Arkansas, has been apointed assistant professor of mathematics at Lehigh University.

Associate Professor O.H. Rechard, of the University of Wyoming, has been promoted to a professorship of mathematics.

Associate Professor J. F. Ritt, of Columbia University, has been promoted to a professorship of mathematics.

Rafael Sanchez-Diaz, of the College of Agricultural and Mechanical Arts, University of Porto Rico, has been promoted to an assistant professorship.

Assistant Professor Hazel E. Schoonmaker, of the New Jersey College for women, has been appointed professor of mathematics at Hartwick College, Oneonta.

Assistant Professor I. W. Smith, of North Dakota State College, has been promoted to a professorship of mathematics.

Associate Professor R. G. Smith, of Kansas State Teachers College, Pittsburg, has been promoted to a professorship.

Dr. G. W. Starcher, of Ohio University, has been promoted to an assistant professorship of mathematics.

Assistant Professor J. S. Taylor, of the University of Pittsburgh, has been promoted to a professorship of mathematics.

Assistant Professor V. B. Teach, of the Armour Institute of Technology, has been promoted to an associate professorship of mathematics.

Dr. A. Marie Whelan has been promoted to an assistant professorship of mathematics at Hunter College.

Associate Professor F. L. Wren has been promoted to a professorship of the teaching of mathematics at the George Peabody College for Teachers.

Dr. S. D. Zeldin has been promoted to an assistant professorship of mathematics at the Massachusetts Institute of Technology.

The following appointments to instructorships in mathematics are announced:
Brooklyn College, Miss Margaret M. Young;
Brown University, Mr. Max Astrachan;
University of Chicago, Dr. C. W. Mendel, Dr. W. T. Reid;
Columbia University, Dr. A. C. Berry;
Cornell University, Mr. A. H. Black;
Hunter College, Miss Polly P. Nelson;
Long Island University, Mr. K. G. Fuller;
Marquette University, Mr. Stephen Lewandowski;
Princeton University, Mr. J. L. Vanderslice;
Stanford University, Mr. H. M. Bacon;
Trinity College, Mr. W. H. Mitchell;
Wells College, Miss Nancy Cole.

Professor R. K. Butchart, of Raffles College, Singapore, died March 30, 1931.

Professor E. P. Culverwell, of Trinity College, Dublin, died April 17, 1931, at the age of seventy-five.

Professor J. E. Edwards, principal and professor of mathematics and physics at Queen's College, London, died May 16, 1931, at the age of seventy-seven.

Dr. Isidor Fröhlich, emeritus professor of theoretical physics at the University of Budapest, is dead.

Dr. Lucius Hanni, of the University of Graz, is dead.

Professor R. S. Heath, of the University of Birmingham, died April 15, 1931, at the age of seventy-two.

Dr. Eugen Meyer, professor of mechanics at the Berlin Technical School' died December 31, 1930, at the age of sixty-one.

Mr. Norifumi Okamoto, known for his work in the history of Japanese mathematics, died February 17, 1931.

The death is announced of Dr. Wilhelm Valentiner, professor of astronomy at the University of Heidelberg.

Professor Aurel Voss, of the University of Munich, is dead.

Canon J. M. Wilson, of Petersfield, formerly lecturer in mathematics at Cambridge, died April 15, 1931, at the age of ninety-four.

Mr. Edward Bright, of Sandwich, Cape Cod, died March 17, 1931. Mr. Bright had been a member of the American Mathematical Society since 1926.

Professor I. P. Church, emeritus professor of civil engineering at Cornell University, died May 7, 1931, at the age of eighty.

Dr. A. A. Himwich, a physician of New York City, died April 18, 1931, at the age of sixty-nine. Dr. Himwich had been a member of the American Mathematical Society since 1898.

Professor A. A. Michelson, of the University of Chicago, the distinguished physicist, died May 9, 1931, at the age of seventy-eight.

Professor W. A. Zehring, of Purdue University, died May 1, 1931, at the age of fifty-five. Professor Zehring had been a member of the American Mathematical Society since 1911.

ABSTRACTS OF PAPERS

SUBMITTED FOR PRESENTATION TO THIS SOCIETY

The following papers have been submitted to the Secretary and the Associate Secretaries of the Society for presentation at meetings of the Society. They are numbered serially throughout this volume. Cross-references to them in the reports of the meetings will give the number of this volume, the number of this issue, and the serial number of the abstract.

235. Professor Clifford Bell: *Useful functions associated with rational cubic curves.*

The six parameters of the intersection and contact points of a rational plane cubic with a circumscribing triangle of reference appear in the parametric equations of the cubic. A certain function of these parameters, invariant under any linear transformation on the parameter of the cubic, is shown to have some useful properties for determining the nature of the singularity of the curve. Like results are obtained when an inscribed triangle is used for the triangle of reference. (Received May 11, 1931.)

236. Professor R. L. Wilder: *A converse of the theorem regarding the separation of E_3 by a closed two-dimensional manifold of genus p.* Second paper.

It is shown that a bounded point set which is a common boundary, in E_3, of two (this does not imply *only* two) uniformly connected im kleinen domains D_1 and D_2 such that the connectivity number $R^1(D_1)$ is finite, is a closed two-dimensional manifold. This theorem still holds if the condition that $R^1(D_1)$ be finite is replaced by the condition that there is a positive number ϵ such that no 1-cycle of $D_i (i=1, 2)$ of diameter less than ϵ links K. (Received May 11, 1931.)

237. Professor C. M. Cramlet: *An interpretation of the invariants of a system of linear homogeneous second-order differential equations.*

A complete system of tensors and invariants of the system of equations $d^2y^i/dx^2+p_\alpha{}^i(x)dy^\alpha/dx+q_\alpha{}^i(x)y^\alpha=0$ were obtained in an earlier paper. Here new tensors q_{rs} are obtained which establish a metric at every point of an integral curve. The notion of parallel displacement is introduced in an invariantive manner so that comparisons of directions can be made at different points of an integral curve of the basic differential equations. (Received May 13, 1931.)

238. Professor R. M. Winger: *On certain projective trochoids.*

In this paper the author considers two pencils of rational curves of order $2n$ which are invariant under dihedral groups of order $2(2n-1)$ and $4(n-1)$ respectively. The treatment is general, properties being stated for arbitrary n. In the proper metrical setting, the curves are trochoidal, exhibiting multiple symmetry. While both pencils fall short of the maximum symmetry for algebraic curves, the first possesses the greatest symmetry possible for rational trochoids. (Received May 13, 1931.)

239. Professor J. P. Ballantine. *The numerical solution of linear equations by vectors.*

The notion of vectors leads to a very simple and rapid method of solving linear equations. No errors of rounding off, which by other methods result from multiplications of decimals, are possible. Errors due to mistakes can be checked at any step. Answers of increasing accuracy can be read off at each step. (Received May 14, 1931.)

240. Professor A. R. Jerbert: *A certain congruence associated with a ruled surface.*

Following Wilczynski let y and z denote the points $y_i(t)$, $z_i(t)$, $(i=1, 2, 3, 4)$, which generate the flecnode curves of a general ruled surface. It is evident then, that the point $y+\lambda z$ ($\lambda=$const.) generates a unique one-parameter family of curves on the ruled surface. The tangents to these curves constitute a congruence of lines whose relation to the surface is developed in detail. (Received May 14, 1931.)

241. Professor W. E. Milne: *On the numerical integration of certain differential equations of the second order.*

Differential equations of the second order in which the first derivatives do not appear are of frequent occurrence and great importance in many applied fields. For their numerical integration a special set of quadrature formulas of marked simplicity is derived. The method of solution is well adapted for use with a calculating machine, provides a check on each step, and requires a minimum of written calculation. (Received May 15, 1931.)

242. Professor R. E. Moritz: *A new theory of depreciation of physical assets.*

The various methods employed for the computation of depreciation of physical assets are based on the assumption that the life time of the asset is a known constant. In this paper it is shown that this is not true, but that the life time is a function of several variables one of which is the assumed rate of depreciation. In this paper depreciation is treated from the point of view here suggested. A method for determining the life time of an asset is developed such that the annual charge to production necessary to recover original or replacement costs and all repair charges shall be a minimum. (Received May 15, 1931.)

243. Professor R. E. Moritz: *On a totally discontinuous function.*

In this paper it is shown that every continuous function of x may be defined as the limiting case of a function whose derivative fluctuates between $+\infty$ and $-\infty$ in every, however small, interval of x. (Received May 15, 1931.)

244. Dr. Leo Zippin: *On the Moore-Kline problem.*

By a Moore-Kline set the author denotes a closed compact point set M whose maximal connected subsets are arcs or points, and such that no inner point of an arc L of M is a limit point of $M-L$; by a generalized continuous curve, a separable, metric, complete, connected, and locally connected space. It is proved in this paper that, in order that a generalized continuous curve C contain, for every one of its Moore-Kline subsets M, an arc in which M is embedded, *it is necessary and sufficient* that (A) if X is any arc of C, and t one of its end points, then for every $e>0$ there exists a $d>0$ such that if x and y are points of $S(t, d)\cdot(C-X)$, the points x and y may be arc-joined in $S(t, e)\cdot(C-X)$. This property A is shown to have the following non-metric equivalent: (B) if D is an open connected subset of C, and ab an arc lying in D except for the point a which belongs to the boundary of D, then $D-(ab-a)$ is connected. In this form, it will be recognized as an early plane axiom of R. L. Moore. The author gives also an extension to generalized continuous curves of a theorem recently announced by G. T. Whyburn (see Abstract No. 36–9–329), with an independent proof which is derived by the methods employed in the Moore-Kline problem. (For special cases of this problem, compare E. W. Miller, Abstract 36–5–265.) (Received May 15, 1931.)

245. Dr. I. J. Schoenberg: *Some applications of the calculus of variations to Riemannian geometry.*

In 1855 O. Bonnet, using Jacobi's variational equation for geodesics on a surface and Sturm's comparison theorem for linear second-order differential equations, established a theorem relating the curvature of an ovaloid surface to its size (see W. Blaschke, *Vorlesungen über Differentialgeometrie*, vol. 1, 3d ed.). The author gives here a comparison theorem for the second variation in the calculus of variations, which, together with Levi-Civita's system of variational equations for geodesics of a Riemannian manifold V_n (see T. Levi-Civita, *Sur l'écart géodesique*, Mathematische Annalen, vol. 97), permits one to generalize O. Bonnet's theorem to a manifold V_n of positive curvature, satisfying presupposed conditions concerning the existence of the absolute shortest distance between two of its points. The problem of the shortest distance in a V_n is also discussed by means of a sufficiency theorem of Bliss for the general parametric problem in the calculus of variations. (Received May 16, 1931).

246. Professor G. A. Bliss and Dr. I. J. Schoenberg: *On separation, comparison and oscillation theorems for self-adjoint systems of second-order linear differential equations.*

This paper gives a generalization of the Sturm-Liouville separation, comparison and oscillation theorems for one second-order linear differential equation, which is based on what we call a conjugate system of points with respect to a self-adjoint system of second-order linear differential equations. Such a conjugate system of points is a generalization of the set of zeros of a solution of a second-order linear differential equation. Our separation theorem includes results of J. Radon on conjugate points in the calculus of variations (see J. Radon, *Zum Problem von Lagrange*, Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg, vol. 6, pp. 298–299). The comparison theorem, proved by means of some arguments of the calculus of variations, makes it possible to extend to this more general situation the classical continuity proof for the Sturm-Liouville oscillation theorem (see L. Bieberbach, *Differentialgleichungen*, 2d ed.). (Received May 16, 1931.)

247. Professor A. D. Michal: *Concerning a mixed functional equation and an associated invariant theory.*

The object of this paper is twofold: first, to develop the properties of the solutions of the mixed functional equation (1) $\partial A_i(\gamma, x)/\partial x = L[A_{i-1}(x, s); \gamma]$ and second, to study the $A_i(\tau, x)$ that are solutions of (1) and are in addition relative invariants, with a factor a^{i-s}, under the mixed point and functional transformation $\bar{x}=ax+b$, $\bar{y}(\tau)=F[y(s); \tau]$. The functionals L and F are assumed to be linear. Linear differential expressions and linear functionals of the Fredholm type furnish interesting examples. In the latter case, the Fredholm integral equation theory plays a fundamental role. (Received May 16, 1931.)

248. Dr. H. C. Ayres: *Some theorems concerning matrices of continuous functions.*

Bliss, in an article in the Transactions of this Society in 1918, has demonstrated the following theorem: If a matrix $\|a_{\mu\nu}\|$ of continuous functions $a_{\mu\nu}(x)$ has rank m at every point in an interval x_1x_2, then an additional $n-m$ rows of polynomials $a_{\tau\nu}(x)$ can always be selected so that the determinant $|a_{\sigma\nu}|$ secured by adding them as rows to the original matrix is non-vanishing in the interval x_1x_2. The method used by Bliss apparently cannot be extended to the n-variable case. The purpose of this paper is to attack the problem by a different method which may be extended to n variables by the use of mathematical induction. (Received May 18, 1931.)

249. Professor L. H. McFarlan: *The problem of minimizing a double integral with adjoined partial differential equations.*

This paper treats the problem of giving a minimum value to a double integral $\iint f(x, y, z_1, \cdots, z_n, z_{1x}, \cdots, z_{nx}, z_{1y}, \cdots, z_{ny})dx\,dy$, where the functions $z_1(x, y), \cdots, z_n(x, y)$ take on assigned values along the boundary of the region of integration and also satisfy a system of $m<n$ partial differential equations of the first order. The first necessary condition on the minimizing functions is stated in the form of a multiplier rule similar to the well known Lagrange multiplier rule for the Problem of Lagrange in one independent variable. The multipliers are functions of both x and y. (Received May 18, 1931.)

250. Dr. C. E. Seely: *Note on kernels of positive type.*

This note gives sufficient conditions for the existence of a characteristic constant for non-symmetric kernels of positive type. (Received May 22, 1931.)

251. Professor G. T. Whyburn: *Concerning continuous images of the interval.*

In this paper there is given a new proof for the celebrated theorem of Hahn-Mazurkiewicz to the effect that *any compact, metric, and locally connected continuum M is the image under a continuous transformation of the unit interval,* which seems to be simpler and more elementary than any previously known proof for this fundamental proposition. The new proof is effected by modifying a proof given by Hahn for the weaker theorem that *any two points of such a continuum M can be joined by a subset M′ of M which is the continuous image of the interval* in such a way as to insure that the subset M' obtained will be identical with M. (Received June 1, 1931.)

252. Professor Karl Menger: *Remarks concerning a paper of W. L. Ayres.*

The author calls attention to some further results suggested by the paper of W. L. Ayres on the regular points of a continuum published in the Transactions of this Society (vol. 33, pp. 252–262), and makes some additions to the bibliography given in that paper. (Received June 12, 1931.)

253. Professor Edward Kasner: *Circular trajectories in dynamics.*

It is known that the ∞^3 circles of the plane do not constitute a family of dynamical trajectories (see Kasner, Transactions of this Society, 1906, or the Princeton Colloquium, 1913). The problem of the present paper is to find (positional) fields of force whose ∞^3 trajectories include ∞^2 circles. All central fields of this sort are determined explicitly. The result contains Maxwell's force (varying as the inverse fifth power) as a special case. (See also the writer's paper in the National Academy Proceedings, June, 1931.) The ∞^3 circles, though not in the dynamical type, are in both the sectional type and the curvature trajectory type. (Received June 12, 1931.)

254. Dr. E. J. McShane (National Research Fellow): *On the semi-continuity of double integrals in the calculus of variations.*

The integrals considered are of the form $F(S)=\int\int f(x, X)du\equiv\int\int f(x^1, x^2, x^3, X^1, X^2, X^3)du^1du^2$, where S is a rectifiable surface, $x^i=x^i(u^1, u^2)$, the functions $x^i(u)$ satisfying a Lipschitz condition, and where the X^i are the Jacobians of the x^i with respect to the u^j. If (1) $f(x, X)\geqq 0$ for every x and every $X\neq(0,0,0)$, and (2) $E(x, X, \overline{X})\geqq 0$ for every x, every $X\neq(0, 0, 0)$, and every $\overline{X}\neq(kX^1, kX^2, kX^3)$, $k\geqq 0$, then, for every constant N, $F(S)$ is lower semi-continuous on the class of all rectifiable surfaces of area $\leqq N$. If in either (1) or (2) the equality be excluded, then $F(S)$ is lower semi-continuous on the class of all rectifiable surfaces. Conditions sufficient for lower semi-continuity at a given surface are also obtained. (Received May 27, 1931.)

255. Professor O. E. Glenn: *Proof of a generalized form of Newton's law of gravity.*

The formula derived in this paper for central gravitational force, for astral motion, results from an hypothesis not previously used in celestial mechanics. A determinate stable central orbit, around which a mass is rotating, is assumed to have a self-restitutional property which restores it when it is perturbed by neighboring astronomical objects. Physical chemists have referred to an analogous property of atomic orbits. This hypothesis alone, developed by invariantive methods under a theory of fields of perturbations, gives, for the central force function, $F=2\gamma^2\epsilon^2[(u/v^2)/r^2+(3u^2/v^2+(1/2)\epsilon^2+1)/r^3+(3u^2/v^2+3u)/r^4+((u^2+v^2)^2/v^2)/r^5]$, r being the distance and the other literal quantities being constant. The Newtonian formula is a special instance. A brief account of the theory, with astronomical interpretations, is to be published by the Indiana Academy of Science. Further investigation is in progress. (Received April 17, 1931.)

256. Mr. E. A. Kholodovsky: *On a generalized binomial series.*

The series in question, $S_{(k,q,n)}^{(\alpha)}=1+nq+C_n^2{}_{-(k-1)}q^2+\cdots+C_{n-l(k-1)}^{l+1}{}^{l+1}q+\cdots$, a generalization of the binomial series (for $k=1$ and $k=0$), plays an important role in certain problems in biology dealing with the multiplication of organic beings and geochemical constants. The author investigates this series with regard to convergence and summability (generalized Euler's method). The formula $S_{(k,q,n)}^{(\alpha)}=e^{\phi_k(q)n+\psi_k(q)}$ is established, where $\phi_k(q)$, $e^{\phi_k(q)}$, and $e^{\psi_k(q)}$ can be written as series of the same character as $S_{(k,q,n)}^{(\alpha)}$. An interesting relation is derived between $S_{(k,q,n)}^{(\alpha)}$ and the trinomial equation $z^k-z^{k-1}-q=0$, which yields a very simple explicit expression for $S_{(k,q,n)}^{(\alpha)}$. Some special cases are considered. (Received March 11, 1931).

257. Mr. E. A. Kholodovsky: *On a method of summation of infinite series and a geometrical interpretation of the method.*

This paper generalizes Euler's method of differences. It considers the summatrix series $\sum_{n=0}^{k}m^{k-n}C_k^n$; the sum is $\lim_{k\to\infty}{}_m\sigma_p^{(k)}=\lim\{1/(m+1)^k\}\sum_{n=0}^{k}m^{k-n}C_k^nS_{p+k}$. We have ${}_m\sigma_p^{(k)}=\{1/(1+m)\}\ (m_m\sigma_p^{(k-1)}+{}_m\sigma_{p+1}^{(k-1)})$. If $a_n=f_{(k-1)}(x)\ m^n$, $f_{(k-1)}(x)$ being an integral function of degree $k-1$, then ${}_m\sigma_p^{(k)}=\text{const.}=s$. A geometric interpretation is given similar to that for $m=1$. If $m=1$ the values of $\sigma_p^{(n)}$ can be computed successively as arithmetic means. When the general term a_n of an alternating series is an integral function of degree $k-1$, $\sigma_p^{(k)}$ is constant for $p=-1, 0, 1, 2, \cdots$; this constant is the sum of the series. Hence we have a formula for computing any Bernoulli number. If $\sigma_p^{(k)}=\text{const.}=s$, the curves $y=S_0(x)$ and $y=S_1(x)$ passing respectively through the points $(0, S_0)$, $(2, S_2)$, $\cdots$, and $(1, S_1)$, $(3, S_3)$, $\cdots$ are symmetrical with respect to the straight line $y=s$. The same is true for the curves $y=\sigma_0^{(k)}(x)$ and $y=\sigma_1^{(k)}(x)$ passing respectively through the points $(0, \sigma_0^{(k)})$, $(2, \sigma_2^{(k)})$, $\cdots$ and $(1, \sigma_1^{(k)})$, $(3, \sigma_3^{(k)})$, $\cdots$ $(k=0, 1, 2, \cdots)$. Hence we have a graphical method of summation of divergent series. If a_n is not an integral function, this method approximates the sum. (Received May 13, 1931)

258. Professor R. L. Wilder: *On Jordan continua that are the common boundaries of two or more domains in E_n.*

Examples are given to show that a Jordan continuum in E_3 may be the common boundary of any finite number or a denumerable infinity of mutually exclusive domains. No simple closed curve disconnects such a continuum, and this property is generalized in a theorem for any number of dimensions. If M is a common boundary of two uniformly connected im kleinen domains in E_3 and there is a point P on M and a positive number d such that all 1-cycles of $D_i \cdot S(P, d)(i=1, 2)$ bound in D_i, then M is the common boundary of only two domains. Thus, if M is the common boundary of at least three uniformly connected im kleinen domains (examples are given of such continua), all of the domains link one another in every neighborhood of any point of M. An example is given, in E_3, of a Jordan continuum whose complement has the same connectivity numbers as the complement of the sphere, and is not disconnected by the omission of any arc; however, it not only fails to satisfy the Jordan Curve Theorem internally (see L. Zippin, American Journal of Mathematics, vol. 52, pp. 340–1), but is not disconnected by the omission of any simple closed curve. (Received May 8, 1931.)

259. Dr. A. B. Brown: *Group invariants and torsion coefficients.*

The paper is largely expository. Two sets of invariants of a finite commutative group, their relation to each other, to the Poincaré numbers of an infinite group with a finite number of generators, and to torsion coefficients in analysis situs are considered. Four brief proofs of the uniqueness of the invariants, three of them developed by men working in analysis situs, are mentioned. These are equivalent to four proofs of the group-theoretic part of the proof of the invariance of the torsion coefficients in analysis situs. The question of isomorphism of quotient-groups of isomorphic sub-groups is considered as well as its bearing on a method for determining the invariants. The torsion coefficients in analysis situs are one set of invariants of a certain finite Abelian group determined topologically by the complex. The possibility of using the other set of invariants as torsion coefficients is discussed. (Received May 21, 1931.)

260. Dr. N. H. McCoy, (National Research Fellow): *On the resultant of a system of forms homogeneous in each of several sets of variables.*

The resultant of a system of general forms, homogeneous in each of several sets of variables, is here shown to have properties analogous to the well known properties of the resultant in the case of a single set of variables. Following Koenig (Algebraischen Grössen), it is shown that the resultant is determined by the two properties (i) that it is an irreducible polynomial in the coefficients of the general forms, and (ii) that it belongs to the module defined by the forms. In particular, a general expression is obtained for the degrees and weights of the resultant, thus furnishing a proof of a theorem stated by Sylvester. It is then found that Sylvester's dyalitic method of elimination may be

extended to the case of certain forms of the type here discussed. As special cases it is shown that the method may be used to obtain the resultant of multiple binary forms with certain restrictions on the degrees. The resultant of a system of forms linear in each of the sets of variables may also be expressed in determinant form by this method. (Received May 25, 1931.)

261. Dr. R P. Agnew (National Research Fellow): *On deferred Cesàro means.*

With each pair of sequences $\{p_n\}$ and $\{q_n\}$ of non-negative integers satisfying the conditions $q_n > p_n$, $n = 1, 2, 3, \cdots$, and $\lim_{n\to\infty} q_n = +\infty$, we associate a method $D(\{p_n\}, \{q_n\})$ of summability which assigns to a given sequence $\{s_n\}$ the value $\lim_{n\to\infty} (s_{p_n+1}+s_{p_n+2}+\cdots+s_{q_n})/(q_n-p_n)$ when this limit exists. In case $\lim_{n\to\infty} p_n = +\infty$, $D(\{p_n\}, \{q_n\})$ has useful properties not possessed by (M), the Cesàro arithmetic mean method. Methods of the form $D(\{p_n\}, \{q_n\})$ are compared in a study of mutual consistency, relative inclusion, and equivalence. (Received June 8, 1931.)

262. Dr. L. M. Blumenthal: *Note on Volterra and Fredholm products of symmetric kernels.*

This paper establishes two theorems concerning Volterra and Fredholm products of two continuous symmetric functions. Theorem 1: If the Volterra product of two continuous symmetric functions is a symmetric function, then one of the functions is identically zero. Theorem 2: In order that the Fredholm product of two continuous symmetric kernels, admitting only a finite number of characteristic values, be symmetric, it is necessary and sufficient that the characteristic functions of the two kernels be orthogonal, or that $\phi_i(x) = c_{ij}\theta_j(x)$ where $\phi_i(x)$ and $\theta_j(x)$ are the characteristic functions of the kernels and c_{ij} is a constant. The first theorem is proved by first showing that two symmetric functions have a symmetric product (Volterra) if and only if the two functions are skew permutable; that is, $K_1K_2 = -K_2K_1$. It is then shown that if a continuous function is skew permutable with a given continuous function of determined order, the function is identically zero. Though the second theorem can be discussed in terms of permutability of the second kind (as is shown), it is obtained by special methods more germane to the problem considered. (Received June 16, 1931.)

263. Dr. L. M. Blumenthal: *An application of metric geometry to determinants.*

This paper points out by means of examples that a certain theorem on determinants announced by H. W. Richmond (Proceedings of the Cambridge Philosophical Society, vol. 14 (1908), p. 475) and which formed the basis of a paper by B. Segre (Atti, Accademia dei Lincei, (6), vol. 2 (1925), p. 539) is not valid. The main interest of the paper is the establishment of the following "counter-theorem": If a symmetric determinant of the fifth order $|a_{ij}|$, $a_{ij} = a_{ji}$, $a_{ij} = 1$, $a_{ij} > 0 (i \neq j)$, $a_{ii} = 0$, $(i, j = 1, 2, 3, 4, 5)$ is different from zero, and the complementary minors of four of the elements in the principal diagonal vanish, then the complementary minor of the remaining element does not vanish. This is counter to the above-mentioned "theorem" of Richmond-Segre, for de-

terminants of order five. The determinant contains six distinct positive elements different from unity. Interpreting these numbers as mutual distances of four distinct points, it is shown that the four points are pseudo-linear (Karl Menger, *Untersuchungen über allgemeine Metrik*, Mathematische Annalen, vol. 100 (1928), p. 125) and the theorem follows from known properties of these points. Four corollaries give interesting properties of the class of determinants dealt with in the theorem. (Received June 16, 1931.)

264. Professor R. G. Lubben: *Two theorems concerning abstract spaces.*

(1) In order that in a space H Fréchet each point set either contain a point of condensation of itself or be separable, it is necessary and sufficient that every point set contain a limit point of itself. (2) A space S Fréchet or a Hausdorff space which satisfies the first countability axiom and is locally compact is regular. Theorem (1) may be extended to more general spaces. (Received June 19, 1931.)

265. Miss Beatrice Aitchison: *Concerning regular accessibility.*

This paper comprises a systematic treatment of the subject of regular accessibility from a new point of departure, suggested by the recently established theorem of Moore-Menger that any connected, locally connected G_δ set is arcwise connected. A general theorem on the regular accessibility of limit points of such sets is proved, and most of the known propositions on regular accessibility are developed from this point. Simpler proofs are obtained, and some of the previously known results have been generalized substantially. (Received June 23, 1931.)

266. Professor B. W. Jones: *On Selling's reduction of positive ternary quadratic forms.*

In a positive ternary quadratic form f, L. Charve (Annales de L'Ecole Normale, (2), vol. 9 (1880), Suppl.) replaces the variables x, y, z by $x-t, y-t, z-t$ respectively to give the form $\phi = -g(y-z)^2 - h(z-x)^2 - k(x-y)^2 - l(x-t)^2 - m(y-t)^2 - n(z-t)^2$. With Selling, he calls f reduced if none of the coefficients of ϕ are negative and proves algebraically that the ϕ of such a reduced form is unique except for permutations of g, h, k, l, m, n. This paper provides further inequalities on the coefficients which result in a unique form f. This theory of reduction is correlated with the Eisenstein method of reduction first by showing that every Eisenstein reduced form with non-positive cross-product coefficients coincides with the reduced form of this paper and second, by finding the transformations that take the reduced forms of one type into those of the other, thus giving an algebraic proof of Eisenstein's reduction. (Received June 29, 1931.)

267. Professor A. A. Albert: *A note on cyclic algebras of order sixteen.*

In this note it is proved that there exist cyclic normal division algebras of order sixteen over a field $R(x, y)$, x and y indeterminates, such that the algebras do not satisfy the Wedderburn norm condition for cyclic algebras and are

direct products of generalized quaternion algebras. This example is a modification of an example of R. Brauer in that he also considered division algebras which were such direct products over such a field but these algebras are cyclic and moreover the proof that these algebras are division algebras is essentially different from his. (Received June 29, 1931.)

268. Professor A. A. Albert: *On the construction of cyclic algebras with a given exponent.*

The problem of the construction of cyclic algebras has been reduced to the case where the order of the algebra is a power of a single prime. The author then considers cyclic algebras of order p^{2e}, p a prime, generated by a cyclic field Z or order p^e over the general non-modular field F, and a number γ in F. It is proved that the least power of γ which is the norm of an element of Z is a power of p and that this integer is the exponent of the algebra. The field $Z=Z_e$ contains sub-fields $Z_{e-1}, \cdots, Z_1, Z_0=F$ of orders p^i respectively and it is proved that γ^α, $\alpha=p^i$ is the norm of a quantity of Z if and only if γ itself is the norm $N_i(f)$ for the field Z_i, of a quantity of Z_i. This criterion is used to give a particularly simple criterion for the construction of all cyclic algebras over an algebraic field of finite order with respect to the field of all rational numbers. (Received June 29, 1931.)

269. Professor A. A. Albert: *Division algebras over an algebraic field.*

The theory of quadratic forms over an algebraic field $R(x)$ of H. Hasse, (Crelle, 1923) applied to give extensions of the theorems of the author's recent papers on rational division algebras to algebras over $R(x)$. In particular it is proved that the direct product of two generalized quaternion algebras over $R(x)$ is never a division algebra, that a necessary and sufficient condition that a normal division algebra of order sixteen over $R(x)$ be a cyclic (Dickson) algebra is that the algebra contain a quantity u not in $R(x)$ but such that $u^2=a^2+b^2$ where a and b are in $R(x)$. The theorems of an as yet unpublished Transactions paper of Hasse, *The theory of cyclic algebras over an algebraic number field*, are applied to give an alternative proof of the above theorem on generalized quaternion algebras and it is in fact proved that a necessary and sufficient condition that a direct product of two normal division algebras over $R(x)$ be a division algebra is that their orders be relatively prime. (Received June 29, 1931.)

270. Mr. C. H. Harry: *An examination of some cut sets of space.*

This paper treats some cut sets of a connected, locally connected, locally compact and metric space S. Using the property that any collection (X) of closed and non-separated cuttings of S which separate any two fixed points a and b is ordered, the paper proves that any infinite monotonic collection (X_i) of sets X is convergent and has a non-vacuous limit L which separates a and b if $L\cdot(a+b)=0$. The second part of the paper deals with the collection L of all points x which together with some point y separate two fixed points a and b. The theorem established is that $L+a+b$ is closed and compact. (Received July 1, 1931.)

271. Professor Morris Marden: *On Stieltjes polynomials* (Second Paper.)

The author has simplified his previous treatment (see Abstract No. 36–9–341) through introducing the idea of "covering function" of a convex region, by which he means a function $k(\lambda)$ such that the inequality $|z-\lambda| \leqq k(\lambda)$ is satisfied by and only by points z in K. The result that, if the a_j lie in a circle C of radius r, the zeros of every characteristic and Stieltjes polynomial lie in a concentric circle C' of radius $r\sec\gamma$ has been thereby extended to read that, if the a_j lie in a convex region K with covering function $k(\lambda)$, these zeros lie in a convex region K' with covering function $k(\lambda)\sec\gamma$. If, for instance, K is an ellipse with major axis A, K' is a confocal ellipse of major axis $A\sec\gamma$. The paper also considers the case that the coefficient of dw/dz in the differential equation is the ratio of two real polynomials. The results are somewhat more extensive and the proof much simpler than the corresponding in the doctoral thesis of Charles Vuille, of Zurich. (Received July 3, 1931.)

272. Dr. M. G. Scherberg: *The Degree of Convergence of a Series of Bessel Functions.*

The author finds the degree of convergence; i.e., the order of magnitude of the difference between a function $f(x)$ and the first n terms of a series representing it, of the Bessel Series known as the Fourier-Bessel Series and also the Dini Series of Bessel Functions. In the first part of the paper $f'(x)$ is restricted to limited variation and permitted of having a finite number of finite jumps. The results are reported in the set of five theorems which show the influence of the conditions at the end points of the interval of convergence. The second part allows $f(x)$ $(p-1)$ continuous derivatives and a pth derivative like that of $f'(x)$ above. A set of theorems analogous to those in the previous section are found. The third and final part of the paper deals with the order of magnitude of the constants which are found in formulas employed in the development of the first two sections of the paper. Thus the results of the paper may readily be carried through in numerical detail for any particular problem in hand. (Received July 6, 1931.)

273. Mr. Max Coral: *The Euler-Lagrange multiplier rule for double integrals.*

Let $z_i=Z_i(x, y)(i=1, 2)$ be a surface of class C'' furnishing a minimum for $I=\iint f(x, y, z_i, p_i, q_i)\,dxdy$ $(p_i=\delta z_i/\delta x,\ q_i=\delta z_i/\delta y)$ in a class of surfaces $z_i=z_i(x, y)$ which have the same boundary above the unit square B in the (x, y)-plane and which satisfy the differential equation $\phi(x, y, z_1, z_2, p_1, p_2)=0$. Gross (Monatshefte für Mathematik und Physik, vol. 27 (1916), pp. 114–120) showed that if the minimizing surface is "normal," there exists a unique function $\lambda(x, y)$ of class C' in B such that along the minimizing surface the Euler-Lagrange differential equations $F_{z_i}-(\delta/\delta x)F_{p_i}-(\delta/\delta y)F_{q_i}=0$, formed for the function $F=f+\lambda\phi$, are satisfied. In the present paper a somewhat more general region B is considered, an improved definition of normality is given, and a much simpler proof is made of the multiplier rule for the above problem of Lagrange for double integrals by introducing an auxiliary problem of Lagrange for simple integrals. Furthermore, similar methods are found to be effective in

proving the multiplier rule for the case when the equation of condition has the more general form $\phi(x, y, z_1, z_2, p_1, p_2, q_2)=0$. (Received July 7, 1931.)

274. Dr. I. J. Schoenberg (International Research Fellow): *A supplement to the law of inertia for real quadratic forms with an application to the projective geometry of hyperquadrics.*

In this paper the following results are established: 1. Let the real quadratic form $Q=\sum_1^n a_{ik}x_ix_k$, whose positive and negative type numbers are p and q, be transformed by the real linear transformation (1) $x_i=\sum_{\alpha=1}^m c_{i\alpha}y_\alpha (i=1, 2, \cdots, n)$ of rank r into the new quadratic form $R=\sum_1^m b_{\alpha\beta}y_\alpha y_\beta$ with the type numbers p' and q'. Then the following inequalities hold (2) $p'+q'\leqq r$, $p+q\leqq n$, $0\leqq p-p'\leqq n-r$, $0\leqq q-q'\leqq n-r$. 2. Conservely: If $Q=\sum_1^n a_{ik}x_ix_k$ and $R=\sum_1^m b_{\alpha\beta}y_\alpha y_\beta$ are two real quadratic forms with $p\geqq p'$ and $q\geqq q'$, and if r satisfies the conditions $p'+q'\leqq r\leqq Min(n-p+p', n-q+q', m)$, then there exists a real transformation (1) of rank r carrying Q into R. Theorem 1 yields as a special case $m=n=r$ the law of inertia and theorem 2 gives the conditions of real equivalence of real quadratic forms ($p=p'$, $q=q'$, $n=m$ give $p+q\leqq r\leqq n$). Similar theorems are established for hermitian forms and for the complex field. The equation $Q=0$ represents a hyperquadric in projective coordinates in a $(n-1)$-dimensional space. Making $m=r\leqq n-1$, the theorems stated above permit us to characterize completely from the real projective point of view all the possible different sections of our hyperquadric by all the real linear subspaces of dimensions $r-1=1,2,3, \cdots, n-2$. (Received July 8, 1931.)

275. Dr. T. H. Gronwall: *On the theory of potentiometric titration.*

The end point of the titration is determined by an algebraic equation of the fifth degree with coefficients dependent on two parameters. According to the values of the latter, there are either one or three real and positive roots, and the present paper deals with their separation and approximate computation. (Received July 10, 1931.)

276. Dr. T. H. Gronwall: *An inequality for the Bessel functions of the first kind with imaginary argument.*

In a problem in wave mechanics, it was found necessary to obtain fairly close lower and upper bounds for the expression $y_\gamma(z)=zI_\gamma'(z)/I_\gamma(z)$, the relative order of magnitude of z and γ being unrestricted. The present note accomplishes this by means of the Riccati equation satisfied by $y_\gamma(z)$. (Received July 10, 1931.)

277. Dr. T. H. Gronwall: *A special conformal space of three dimensions occurring in wave mechanics.*

The $ds^2=g_{ij}dx^idx_j$ for which $g^{11}=g^{22}=1$, $g^{33}=2$, $g^{12}=g^{23}=0$, $g^{13}=[(x^1)^2-(x^2)^2+(x^3)^2]/2x^1x^3$, $g^{23}=[(x^2)^2-(x^1)^2+(x^3)^2]/2x^2x^3$, occurs in connection with the wave equation of the helium atom (Hylleraas, Zeitschrift für Physik, vol. 54 (1929), pp. 347–366). This ds^2 is shown to be conformal to a flat ds^2, and the transformation formulas are worked out. The physical application of this transformation forms the subject of another paper. (Received July 10, 1931.)

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OFFICIAL COMMUNICATIONS

Meetings of the Society have been fixed at the following times and places:

MINNEAPOLIS, MINNESOTA, SUMMER MEETING AND COLLOQUIUM, September 8-11, 1931.

NEW YORK CITY, October 31, 1931.

Abstracts must be in the hands of Associate Secretary Tomlinson Fort, 501 West 116th Street, New York City, not later than October 10. In order to be printed in the Bulletin in advance of the meeting, abstracts must be in the hands of the Associate Secretary not later than September 7. By invitation of the program committee, Professor D. V. Widder will deliver an address on *Some recent developments in the theory of algebras*. Professor E. Landau will speak on *Schnirelmann's Theorem*.

COLUMBUS, OHIO, November 27-28, 1931.

Abstracts must be in the hands of Associate Secretary Mark H. Ingraham, University of Wisconsin, Madison, Wisconsin, not later than November 5. All abstracts received by that date will appear in the November issue of this Bulletin.

CALIFORNIA INSTITUTE OF TECHNOLOGY, PASADENA, November 28, 1931.

Abstracts must be in the hands of Associate Secretary T. M. Putnam, University of California, Berkeley, California, not later than October 31. All abstracts received by that date will appear in the November issue of this Bulletin.

NEW ORLEANS, LOUISIANA, ANNUAL MEETING, December 28-31, 1931.

Abstracts are due by November 28, 1931.

The usual February Meeting in New York City will be omitted.

NEW YORK CITY, March 25-26, 1932.

Abstracts are due by March 5, 1932.

STANFORD UNIVERSITY, CALIFORNIA, April 9, 1932.

Abstracts are due by March 12, 1932.

R. G. D. RICHARDSON, *Secretary of the Society.*

Articles for insertion in the BULLETIN should be addressed to E. R. HEDRICK, Editor of the BULLETIN, University of California at Los Angeles. Reviews should be sent to W. R. LONGLEY, Yale University, New Haven, Conn. Notes should be sent to H. W. KUHN, Ohio State University, Columbus, Ohio.

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